The broken-ray transform and its generalizations

Gaik Ambartsoumian

University of Texas at Arlington

100 Years of the Radon Transform

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Acknowledgements

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  - Rim Gouia-Zarrad, American University of Sharjah, UAE
  - Mohammad Latifi-Jebelli, University of Texas at Arlington
  - Sunghwan Moon, UNIST, Korea
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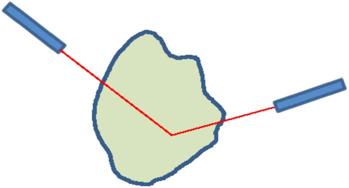
Outline

- Some Motivating Imaging Modalities
- Prior Work and Terminology
- BRT in a Disc
- BRT in a Slab Geometry
- CRT in a Slab Geometry
- Partial Order and Positive Cones
- Cone Differentiation
- BRT in a Slab Geometry (revisited)
- Polyhedral CRT in a Slab Geometry
- Radon-Nikodym Theorem and Range Description

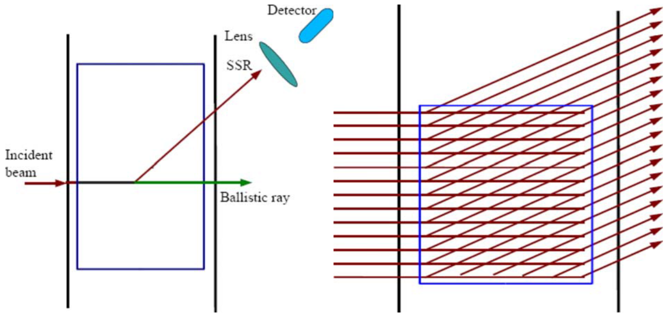
Single Scattering Optical Tomography (SSOT)



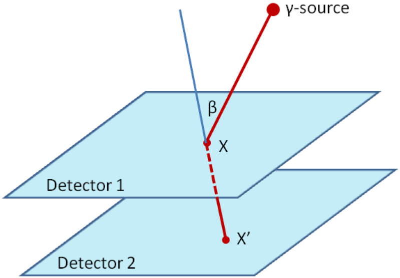
- Uses light, transmitted and scattered through an object, to determine the interior features of that object.
- If the object has moderate optical thickness it is reasonable to assume the majority of photons scatter once.
- Using collimated emitters/receivers one can measure the intensity of light scattered along various broken rays.
- Need to recover the spatially varying coefficients of light absorption and/or light scattering.

Florescu, Schotland and Markel (2009, 2010, 2011)

So if the scattering coefficient is known, then the reconstruction of the absorption coefficient is reduced to inversion of a generalized Radon transform integrating along the broken rays.



Compton Scattering



$$\cos\beta = 1 - \frac{mc^2\,\Delta E}{(E - \Delta E)E}$$

- R. Basko, G. Zeng, and G. Gullberg (1997, 1998)
- M. Nguyen, T. Truong, et al (2000's)

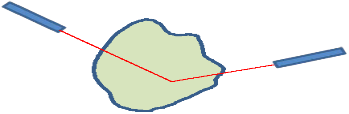
Broken-Ray and Conical Transforms

- L. Florescu, V. Markel, J. Schotland, F. Zhao
- P. Grangeat, M. Morvidone, M. Nguyen, R. Régnier, T. Truong, H. Zaidi
- A. Katsevich, R. Krylov
- F. Terzioglu
- V. Palamodov
- R. Gouia-Zarrad
- M. Courdurier, F. Monard, A. Osses and F. Romero
- D. Finch, B. Sherson

Broken-Ray and Conical Transforms

- M. Cree, P. Bones and R. Basko, G. Zeng, G. Gullberg
- M. Allmaras, D. Darrow, Y. Hristova, G. Kanschat, P. Kuchment
- M. Haltmeier
- C. Jung, S. Moon
- V. Maxim
- D. Schiefeneder
- M. Lassas, M. Salo, G. Uhlmann
- M. Hubenthal, J. Ilmavirta

V-line Radon Transform (VRT) in 2D



Definition

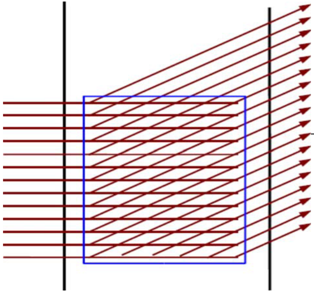
The V-line Radon transform of function $f(x, y)$ is the integral

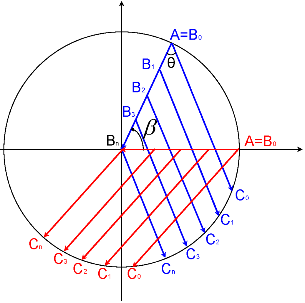
$$\mathcal{R}f(\beta, t) = \int_{BR(\beta,t)} f\, ds, \qquad (1)$$

of f along the broken ray $BR(\beta, t)$ with respect to line measure ds.

The problem of inversion is over-determined, so it is natural to consider a restriction of $\mathcal{R}f$ to a two-dimensional set.

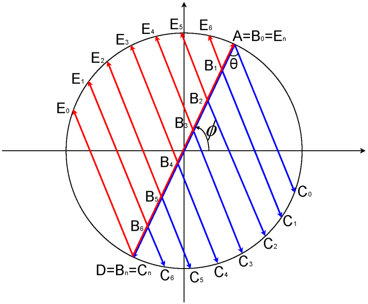
Geometry: Slab vs Disc





- Available directions
- Stability of reconstruction
- Hardware implementation (?)

Full Data



Theorem

If $f(x, y)$ is a smooth function supported in the disc $D(0, R\sin\theta)$, then f is uniquely determined by $\mathcal{R}f(\phi, d)$, $\phi \in [0, 2\pi]$, $d \in [0, 2R]$.

Inversion Formula

$$\widetilde{\mathcal{R}}f(\psi_\phi, t_d) = \mathcal{R}f(\phi, d) + \mathcal{R}f(\phi + \pi, 2R - d) - \mathcal{R}f(\phi, 2R), \quad (2)$$

for all values $\phi \in [0, 2\pi]$ and $d \in [0, 2R]$.

$$f(x, y) = \frac{1}{4\pi} \int_0^{2\pi} \mathcal{H}\left(\widetilde{\mathcal{R}}f\,{}'_t\right)(\psi, x\cos\psi + y\sin\psi)\ d\psi \qquad (3)$$

where $\mathcal{H}$ is the Hilbert transform defined by

$$\mathcal{H}h(t) = -\frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \operatorname{sgn}(r)\,\widehat{h}(r)\ e^{irt}\ dr. \qquad (4)$$

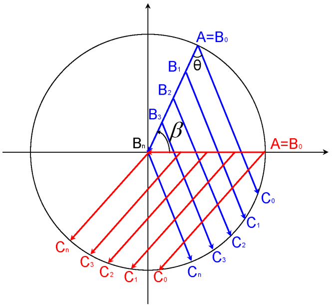
and $\widehat{h}(r)$ is the Fourier transform of $h(t)$, i.e.

$$\widehat{h}(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(t)\ e^{-irt}\ dt. \qquad (5)$$

Inversion Formula

- Issues with the support
- Interior problem
- Other methods without loss of information
- Rotation invariance

VRT in a Disc: Partial Data (G.A., S. Moon 2013)



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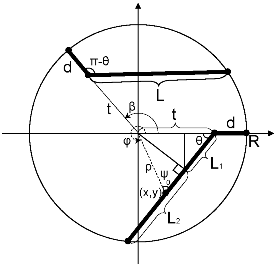
Fourier Expansions



Denote $g(\beta, t) := \mathcal{R}f(\beta, t)$.

$$f(\phi, \rho) = \sum_{n=-\infty}^{\infty} f_n(\rho)\, e^{in\phi}, \qquad g(\beta, t) = \sum_{n=-\infty}^{\infty} g_n(t)\, e^{in\beta},$$

where the Fourier coefficients are given by

$$f_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi, \rho)\, e^{-in\phi} d\phi, \qquad g_n(t) = \frac{1}{2\pi} \int_0^{2\pi} g(\beta, t)\, e^{-in\beta} d\beta.$$

Inversion Formula

$$\mathcal{M}f_n(s) = \frac{\mathcal{M}g_n(s-1)}{1/(s-1) + \mathcal{M}h_n(s-1)}, \quad \Re(s) > 1 \tag{6}$$

where $\mathcal{M}F$ denotes the Mellin transform of function F

$$\mathcal{M}F(s) = \int\limits_0^\infty p^{s-1} F(p)\, dp,$$

and h_n is some fixed function. Hence for any $t > 1$ we have

$$f_n(\rho) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{t-Ti}^{t+Ti} \rho^{-s} \frac{\mathcal{M}g_n(s-1)}{1/(s-1) + \mathcal{M}h_n(s-1)}\, ds. \tag{7}$$

Definition of h_n

If $1 < t < \frac{1}{\sin\theta}$ then

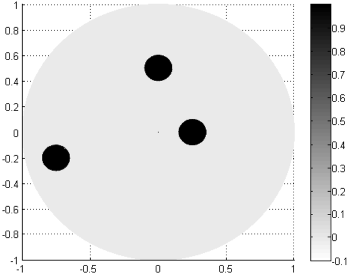
$$h_n(t) = (-1)^n e^{in\psi(t)} \frac{1 + t\cos[\psi(t)] + t^2 \sin[\psi(t)] \frac{\sin\theta}{\sqrt{1 - t^2\sin^2\theta}}}{\sqrt{1 + t^2 + 2t\cos(\psi(t))}}$$

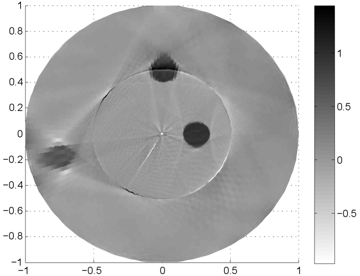
$$-e^{in[2\theta - \psi(t)]} \frac{1 - t\cos[2\theta - \psi(t)] + t^2 \sin[2\theta - \psi(t)] \frac{\sin\theta}{\sqrt{1 - t^2\sin^2\theta}}}{\sqrt{1 + t^2 - 2t\cos[2\theta - \psi(t)]}},$$

$$h_n(t) = (-1)^n e^{in\psi(t)} \frac{1 + t\cos[\psi(t)] + t^2 \sin[\psi(t)] \frac{\sin\theta}{\sqrt{1 - t^2\sin^2\theta}}}{\sqrt{1 + t^2 + 2t\cos[\psi(t)]}}, \quad 0 < t \leq 1$$

and $h_n(t) \equiv 0$, for all $t > \frac{1}{\sin\theta}$. Here $\psi(t) = \arcsin(t\sin\theta) + \theta$.

Numerical Reconstruction (G.A., S. Roy 2015)





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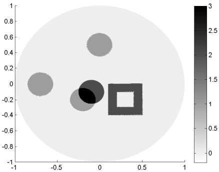
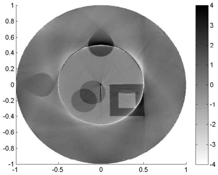
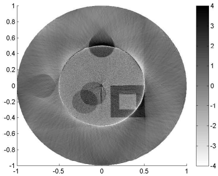


(a) Phantom

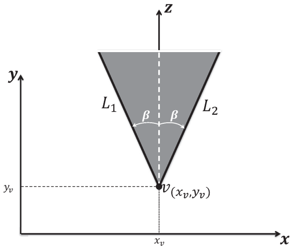


(b) Reconstructed



(c) Reconstruction with 5% multiplicative Gaussian noise.

VRT in Slab Geometry (G.A., R. Gouia-Zarrad 2013)

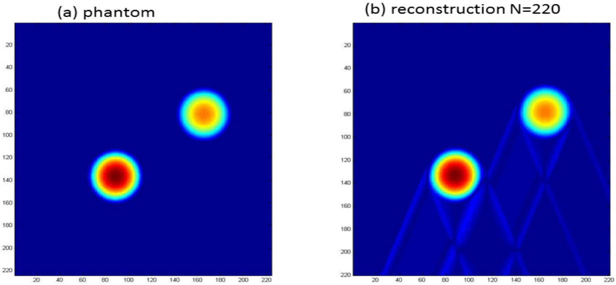


Theorem

$f \in C^\infty(\mathbb{R}^2)$ in $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq x_{max}, 0 \leq y \leq y_{max}\}$. For $(x_v, y_v) \in \mathbb{R}^2$ and fixed $\beta \in (0, \frac{\pi}{2})$ consider the VRT $g(x_v, y_v)$. Then

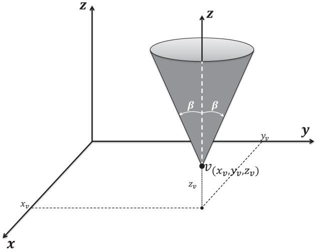
$$f(x, y) = -\frac{\cos\beta}{2}\left(\frac{\partial}{\partial y} g(x, y) + \tan^2(\beta) \int_y^{y_{max}} \frac{\partial^2}{\partial x^2} g(x, t)\, dt\right).$$

Numerical Implementation (2D)



Conical Surfaces of Various Flavors in 3D

CRT in Slab Geometry (G.A., R. Gouia-Zarrad 2013)



Consider a function $f \in C^\infty(\mathbb{R}^3)$ supported in $\mathcal{D} = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x \le x_{max}, 0 \le y \le y_{max}, 0 \le z \le z_{max}\}$. For $(x_v, y_v, z_v) \in \mathbb{R}^3$ we define the 3D conical Radon transform by

$$g(x_v, y_v, z_v) = \int_{C(x_v, y_v, z_v)} f(x, y, z)\, ds.$$

3D Slab Geometry (G.A., R. Gouia-Zarrad 2013)

Theorem

An exact solution of the inversion problem for CRT is given by

$$\widehat{f}_{\lambda,\mu}(z) = C(\beta) \int_{z_{max}}^{z} J_0\left(u(z-x)\right) \left[\frac{d^2}{dx^2} + u^2\right]^2 \int_{z_{max}}^{x} \widehat{g}_{\lambda,\mu}(z_v)\, dz_v\, dx$$

where $\widehat{g}_{\lambda,\mu}(z_v)$ *and* $\widehat{f}_{\lambda,\mu}(z)$ *are the 2D Fourier transforms of the functions* $g(x_v, y_v, z_v)$ *and* $f(x, y, z)$ *with respect to the first two variables,* $C(\beta) = \cos^2\beta/(2\pi\sin\beta)$ *and* $u = \tan\beta\sqrt{\lambda^2+\mu^2}$.

Partial Order in $\mathbb{R}^n$ and Positive Cones

A **Partially Ordered Vector Space** V is a vector space over $\mathbb{R}$ together with a partial order $\leq$ such that

1. if $x \leq y$ then $x + z \leq y + z$ for all $z \in V$
2. if $x \geq 0$ then $cx \geq 0$ for all $c \in \mathbb{R}^+$

From the definition we have $x \leq y \Leftrightarrow 0 \leq y - x$ and hence the order is completely determined by $V^+ = \{x \in V; x \geq 0\}$ **positive cone of** V.

Furthermore, for $P \subset V$ there is a partial order on V such that $P = V^+$ if and only if

$$P \cap (-P) = \{0\}$$

$$P + P \subset P$$

$$c \geq 0 \Rightarrow cP \subset P$$

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Partial Order in $\mathbb{R}^n$ and Negative Cones

We consider partial orders in $\mathbb{R}^n$ corresponding to **negative cones** $(\mathbb{R}^n)^-_{\mathcal{B}}$ generated by a set of fixed basis vectors $\mathcal{B} = \{v_1, ..., v_n\}$, i.e. $(\mathbb{R}^n)^-_{\mathcal{B}} = \{\sum_{i=1}^n c_i v_i; c_i \geq 0\}$.

In $\mathbb{R}^2$ we will use linearly independent vectors u, v as a generating set for the negative cone. In this case the boundary of the negative cone is a *broken line*.

For $f \in L^1(\mathbb{R}^n)$ we define F on $\mathbb{R}^n$ as

$$F(x) = \int_{y \leq x} f(y) d\mu$$

where μ is the Lebesgue measure on $\mathbb{R}^n$ and $y \leq x$ represents the negative cone at x with respect to partial order on $\mathbb{R}^n$.

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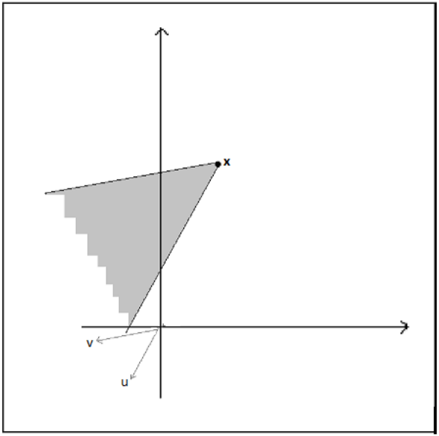

Cone Differentiation Th. (G.A., M. Latifi-Jebelli 2016)

If $\leq$ is the natural order on $\mathbb{R}$, for an integrable function f and

$$F(x) = \int_{y \leq x} f(y)dt$$

we have $F^{'} = f$ almost everywhere. Note that in this case F is absolutely continuous.

Can we have a “similar result” in higher dimensions?

We start from two dimensions. Let f be an integrable function on $\mathbb{R}^2$ (with $\int |f| < \infty$) with respect to Lebesgue measure and define $F(x) = \int_{y \leq x} f(y)d\mu$ using the partial order made by u, v.

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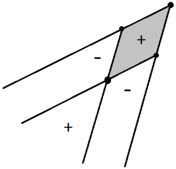
Cone Differentiation Th. (G.A., M. Latifi-Jebelli 2016)

Define $V_{t,s}(x)$ as the average of f over the parallelogram centered at x, sides of length t, s and directions u, v.

Note that the area of the parallelogram made with vectors tu, sv is equal to $|\det(tu, sv)| = ts\,|\det(u, v)|$. Then

$$V_{t,s}(x) = \frac{1}{ts\,|\det(u,v)|}\Big[F(x + \frac{t}{2}u + \frac{s}{2}v) - F(x - \frac{t}{2}u + \frac{s}{2}v) - F(x + \frac{t}{2}u - \frac{s}{2}v) + F(x - \frac{t}{2}u - \frac{s}{2}v)\Big]$$



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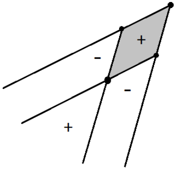

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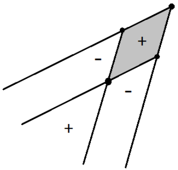

Cone Differentiation Th. (G.A., M. Latifi-Jebelli 2016)

Likewise, for n dimensions by a geometric argument and induction over n we get the following averaging formula for f

$$V_{t_1,\dots,t_n}(x) = \frac{1}{t_1 \dots t_n \left|\det(v_1,\dots,v_n)\right|} \sum_{\alpha \in \{\frac{-1}{2},\frac{1}{2}\}^n} sgn(\alpha_1 \dots \alpha_n) F(x + \alpha_1 t_1 v_1 + \dots \alpha_n t_n v_n)$$

In special case, to get a symmetric neighborhood of x we can let $t_1 = \dots = t_n = t$ to get the average of f over P_t, the parallelograms with sides of length t centered at x

$$V_{t,\dots,t}(x) = \frac{1}{t^n \left|\det(v_1,\dots,v_n)\right|} \int_{P_t} f \, d\mu = \frac{1}{\mu(P_t)} \int_{P_t} f \, d\mu.$$

Cone Differentiation Th. (G.A., M. Latifi-Jebelli 2016)

Likewise, for n dimensions by a geometric argument and induction over n we get the following averaging formula for f

$$V_{t_1,\ldots,t_n}(x) = \frac{1}{t_1 \ldots t_n \left|\det(v_1,\ldots,v_n)\right|} \sum_{\alpha\in\{\frac{-1}{2},\frac{1}{2}\}^n} sgn(\alpha_1 \ldots \alpha_n) F(x + \alpha_1 t_1 v_1 + \ldots \alpha_n t_n v_n)$$

In special case, to get a symmetric neighborhood of x we can let $t_1 = \cdots = t_n = t$ to get the average of f over P_t, the parallelograms with sides of length t centered at x

$$V_{t,\ldots,t}(x) = \frac{1}{t^n \left|\det(v_1,\ldots,v_n)\right|} \int_{P_t} f \, d\mu = \frac{1}{\mu(P_t)} \int_{P_t} f \, d\mu.$$

Cone Differentiation Th. (G.A., M. Latifi-Jebelli 2016)

Now by averaging over this infinitesimal symmetric neighborhood of x and applying the Lebesgue Differentiation Theorem we have

Theorem

Let $\leq$ be an order in $\mathbb{R}^n$ made from the positive cone of $v_1, \ldots, v_n$ and for $f \in L^1(\mathbb{R}^n)$ define

$$F(x) = \int_{y \leq x} f(y) d\mu.$$

Then for almost every x we have

$$f(x) = \lim_{t \to 0} V_{t,\ldots,t}(x).$$

Cone Differentiation Th. (G.A., M. Latifi-Jebelli 2016)

Theorem

Let the hypothesis of the previous theorem be satisfied and f be continuous. Then

$$f(x) = \frac{1}{|\det(v_1, \ldots, v_n)|} \frac{\partial}{\partial v_1} \cdots \frac{\partial}{\partial v_n} F(x),$$

where $\frac{\partial}{\partial v_j}$ denotes the directional derivative along vector v_j.

How does this help us with the inversion of the broken-ray or conical Radon transforms?

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Inversion of BRT in 2D (G.A., M. Latifi-Jebelli 2016)

Assume that $L(x, y)$ is the unique broken ray with vertex at (x, y) and axis of symmetry α, where $\alpha = (\alpha_x, \alpha_y)$ is a unit vector parallel to $\frac{u+v}{2}$. Also, let β be the angle between u and α.

Theorem

Let $\mathcal{R}$ be the broken ray transform on $L^1(\mathbb{R}^2)$ defined by:

$$(\mathcal{R}f)(x, y) = \int_{L(x,y)} f dL.$$

Then

$$F(x, y) = \int_0^\infty (\mathcal{R}f)(x + t\alpha_x, y + t\alpha_y) \sin\beta \, dt$$

is the integral of f over the negative cone at (x, y). Hence

$$f(x, y) = \frac{1}{|\det(u, v)|} \frac{\partial}{\partial u} \frac{\partial}{\partial v} \int_0^\infty (\mathcal{R}f)(x + t\alpha_x, y + t\alpha_y) \sin\beta \, dt$$

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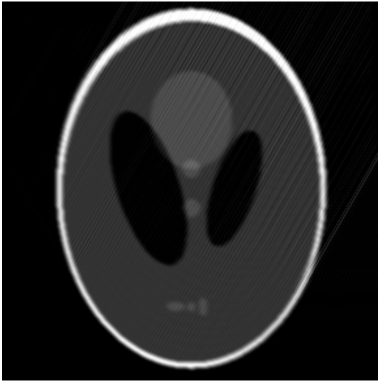

Polyhedral CRT in $\mathbb{R}^n$ (G.A., M. Latifi-Jebelli 2016)

For any $x \in \mathbb{R}^n$, we define $(\mathcal{R}f)(x)$ to be integral over the boundary of polyhedral cone C generated by unit basis vectors $u_1, \ldots, u_n$ starting from x, i.e.

$$(\mathcal{R}f)(x) = \int_{\partial C} f \, dS,$$

where dS is $n-1$ dimensional Lebesgue measure on ∂C.

Assume that $||u_i - u_j||$ is constant for any i and j.

Define $w = \frac{u_1 + \cdots + u_n}{||u_1 + \cdots + u_n||}$.

Let $X_i = span\langle u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n \rangle$ be the hyperplane containing a face of polyhedral cone and define y_i to be a unit vector in $X_i^{\perp}$.

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Polyhedral CRT in $\mathbb{R}^n$ (G.A., M. Latifi-Jebelli 2016)

Theorem

Let $\mathcal{R}, w, y_j$ be defined as above, then

$$F(x) = \int_0^\infty (\mathcal{R}f)(x + wt)\langle w, y_1\rangle \, dt$$

is the integral of f over the polyhedral cone generated by $u_1, \ldots u_n$ starting from x. Hence

$$f(x) = \frac{1}{|\det(v_1, \ldots, v_n)|} \frac{\partial}{\partial v_1} \ldots \frac{\partial}{\partial v_n} \int_0^\infty (\mathcal{R}f)(x + wt)\langle w, y_1\rangle \, dt.$$

Range Description (G.A., M. Latifi-Jebelli 2016)

Existence of f such that $F(x) = \int_{y \leq x} f(y)\, d\mu$.

What is the necessary and sufficient condition for a function F to be a cone integral of another function $f \geq 0$ with respect to a given order structure in $\mathbb{R}^n$?

In case of $n = 1$ the answer was provided by absolute continuity.

We apply the Radon Nikodym Theorem to get the desired description of F. For a given F, we construct a corresponding measure ν that implies existence of f.

We use the above conditions to obtain a range description for CRT.

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Thanks for Your Attention!