Non-abelian Radon transform and its applications

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Consider the equation

$$\theta\partial_x\psi + A(x,\theta)\psi = 0, \quad x \in \mathbb{R}^d,\ \theta \in \mathbb{S}^{d-1}, \tag{1}$$

where A is a sufficiently regular function on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ with sufficient decay as $|x| \to \infty$.
We assume that A and ψ take values in $\mathcal{M}_{n,n}$ that is in $n \times n$ complex matrices.
Consider the "scattering" matrix S for equation (1):

$$S(x,\theta) = \lim_{s\to+\infty} \psi^+(x+s\theta,\theta), \quad (x,\theta) \in T\mathbb{S}^{d-1}, \tag{2}$$

where

$$T\mathbb{S}^{d-1} = \{(x,\theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1} :\ x\theta = 0\} \tag{3}$$

and $\psi^+(x,\theta)$ is the solution of (1) such that

$$\lim_{s\to-\infty} \psi^+(x+s\theta,\theta) = Id, \quad x \in \mathbb{R}^d,\ \theta \in \mathbb{S}^{d-1}. \tag{4}$$

We interpret $T\mathbb{S}^{d-1}$ as the set of all rays in $\mathbb{R}^d$. As a ray γ we understand a straight line with fixed orientation.
If $\gamma = (x, \theta) \in T\mathbb{S}^{d-1}$, then $\gamma = \{y \in \mathbb{R}^d : \; y = x + t\theta, \; t \in \mathbb{R}\}$ (up to orientation) and θ gives the orientation of γ.

We say that S is the non-abelian Radon transform along oriented straight lines (or the non-abelian X-ray transform) of A.

We consider the following inverse problem:

Problem 1. Given S, find A.

Note that S does not determine A uniquely, in general. One of the reasons is that S is a function on $T\mathbb{S}^{d-1}$, whereas A is a function on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ and

$$dim\,\mathbb{R}^d \times \mathbb{S}^{d-1} = 2d-1 > dim\,T\mathbb{S}^{d-1} = 2d-2.$$

In particular, for Problem 1 there are gauge type non-uniqueness, non-uniqueness related with solitons,

and Boman type non-uniqueness.

Equation (1), the "scattering" matrix S and Problem 1 arise, for example, in the following domains:

I. Tomographies:

A. The classical X-ray transmission tomography:

$$n = 1, \quad A(x,\theta) = a(x), \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \tag{5a}$$

$$S(\gamma) = \exp[-Pa(\gamma)], \; Pa(\gamma) = \int\limits_{\mathbb{R}} a(x+s\theta)ds, \; \gamma = (x,\theta) \in T\mathbb{S}^{d-1}, \tag{5b}$$

where a is the X-ray attenuation coefficient of the medium, P is the classical Radon transformation along straight lines (classical ray transformation), $S(\gamma)$ describes the X-ray photograph along γ. In this case, for $d \geq 2$,

$$S\big|_{TS^1(Y)} \ \text{uniquely determines} \ a\big|_Y, \tag{6}$$

where Y is an arbitrary two-dimensional plane in $\mathbb{R}^d$, $TS^1(Y)$ is the set of all oriented straight lines in Y. In addition, this determination can be implemented via the Radon inversion formula for P in dimension $d = 2$.

B. Single-photon emission computed tomography (SPECT):
In SPECT one considers a body containing radioactive isotopes emitting photons. The emission data p in SPECT consist in the radiation measured outside the body by a family of detectors during some fixed time. The basic problem of SPECT consists in finding the distribution f of these isotopes in the body from the emission data p and some a priori information concerning the body. Usually this a priori information consists in the photon attenuation coefficient a in the points of body, where this coefficient is found in advance by the methods of the classical X-ray transmission tomography.

Problem 1 arises as a problem of SPECT in the framework of the following reduction [R.Novikov 2002 a]: $n = 2$,

$$A_{11} = a(x), \quad A_{12} = f(x), \quad A_{21} = 0, \quad A_{22} = 0, \quad x \in \mathbb{R}^d, \qquad (7a)$$

$$S_{11} = \exp\left[-P_0 a\right], \quad S_{12} = -P_a f, \quad S_{21} = 0, \quad S_{22} = 1, \qquad (7b)$$

$$P_a f(\gamma) = \int\limits_{\mathbb{R}} \exp[-Da(x+s\theta,\theta)] f(x+s\theta) ds, \ \gamma = (x,\theta) \in T\mathbb{S}^{d-1},$$

$$Da(x,\theta) = \int\limits_0^{+\infty} a(x+s\theta) ds, \ x \in \mathbb{R}^d, \ \theta \in \mathbb{S}^{d-1}, \qquad (8)$$

where $f \geq 0$ is the density of radioactive isotopes, $a \geq 0$ is the photon attenuation coefficient of the medium, P_a is the attenuated Radon transformation (along oriented straight lines), $P_a f$ describes the expected emission data.

In this case (as well as for the case of the classical X-ray transmission tomography), for $d \geq 2$,

$$S\big|_{TS^1(Y)} \ \text{uniquely determines} \ \ a\big|_Y \ \text{ and } \ f\big|_Y, \tag{9}$$

where Y is an arbitrary two-dimensional plane in $\mathbb{R}^d$, $TS^1(Y)$ is the set of all oriented straight lines in Y. In addition, this determination can be implemented via the following inversion formula [R.Novikov 2002b]:

$$f = P_a^{-1} g, \quad \text{where} \quad g = P_a f,$$

$$P_a^{-1}g(x) = \frac{1}{4\pi}\int\limits_{\mathbb{S}} \theta^{\perp}\partial_x\big(\exp\left[-Da(x,-\theta)\right]\tilde{g}_\theta(\theta^{\perp}x)\big)\,d\theta, \qquad (10a)$$

$$\tilde{g}_\theta(s) = \exp\left(A_\theta(s)\right)\cos\left(B_\theta(s)\right)H(\exp\left(A_\theta\right)\cos\left(B_\theta\right)g_\theta)(s)+$$

$$\exp\left(A_\theta(s)\right)\sin\left(B_\theta(s)\right)H(\exp\left(A_\theta\right)\sin\left(B_\theta\right)g_\theta)(s), \qquad (10b)$$

$$A_\theta(s) = (1/2)P_0 a(s\theta^{\perp},\theta), \quad B_\theta(s) = HA_\theta(s), \quad g_\theta(s) = g(s\theta^{\perp},\theta), \qquad (10c)$$

$$Hu(s) = \frac{1}{\pi}p.v.\int\limits_{\mathbb{R}} \frac{u(t)}{s-t}dt,$$

$$x \in \mathbb{R}^2, \quad \theta^{\perp} = (-\theta_2,\theta_1) \ \text{ for } \ \theta = (\theta_1,\theta_2) \in \mathbb{S}^1, \quad s \in \mathbb{R}.$$

C. Tomographies related with weighted Radon transforms:
We consider the weighted Radon transformations P_W defined by the formula

$$P_W f(x,\theta) = \int\limits_{\mathbb{R}} W(x+s\theta,\theta) f(x+s\theta) ds, \ (x,\theta) \in T\mathbb{S}^{d-1}, \quad (11)$$

where $W = W(x,\theta)$ is the weight, $f = f(x)$ is a test function. We assume that

$$W \in C(\mathbb{R}^d \times \mathbb{S}^{d-1}),$$

$$W = \bar{W}, \quad 0 < c_0 \leq W \leq c_1, \quad (12)$$

$$\lim_{s \to \pm\infty} W(x+s\theta,\theta) = w_{\pm}(x,\theta), \ (x,\theta) \in T\mathbb{S}^{d-1}.$$

If $W = 1$, then P_W is the classical Radon transformation along straight lines.
If

$$W(x,\theta) = \exp\Big(-\int_0^{+\infty} a(x + s\theta)ds\Big),$$

then P_W is the classical attenuated Radon transformation (along oriented straight lines) with the attenuation coefficient $a(x)$.
Transformations P_W with some other weights also arise in applications.
For example, such transformations arise also in fluorescence tomography, optical tomography, positron emission tomography.

The transforms $P_W f$ arise in the framework of the following reduction of the non-abelian Radon transform S: $n = 2$,

$$A_{11} = \theta\partial_x \ln\, W(x,\theta), \quad A_{12} = f(x), \quad A_{21} = 0, \quad A_{22} = 0, \quad (13a)$$

$$S_{11} = \frac{w_-}{w_+}, \quad S_{12} = -\frac{1}{w_+} P_W f, \quad S_{21} = 0, \quad S_{22} = 1. \qquad (13b)$$

For more information on the theory and applications of the transformations P_W; see, for example, [R.Novikov 2014] and [J.Ilmavirta 2016].

D. Neutron polarization tomography (NPT):

In NPT one considers a medium with spatially varying magnetic field.

The polarization data consist in changes of the polarization (spin) between incoming and outcoming neutrons.

The basic problem of NPT consists in finding the magnetic field from the polarization data.

See, e.g., [M.Dawson, I.Manke, N.Kardjilov, A.Hilger, M.Strobl, J.Banhart 2009], [W.Lionheart, N. Desai, S.Schmidt 2015].

Problem 1 arises as a problem of NPT in the framework of the following reduction: $n = 3$,

$$A_{11} = A_{22} = A_{33} = 0, \tag{14}$$

$$A_{12} = -A_{21} = -gB_3(x),\ A_{13} = -A_{31} = gB_2(x),$$

$$A_{23} = -A_{32} = -gB_1(x),$$

where $B = (B_1, B_2, B_3)$ is the magnetic field, g is the gyromagnetic ratio of the neutron.
In this case S on TS^2 uniquely determines B on $\mathbb{R}^3$ as a corollary of Theorem 6.1 of [R.Novikov 2002a]. In addition, the related 3D - reconstruction is based on local 2D - reconstructions based on solving Riemann conjugation problems (going back to [S.Manakov, V.Zakharov 1981]) and on the layer by layer reconstruction approach. The final 3D uniqueness and reconstruction results are global.
For the related 2D global uniqueness see [G.Eskin 2004].

E. Electromagnetic polarization tomography (EPT):
In EPT one considers a medium with zero conductivity, unit magnetic permeability, and small anisotropic perturbation of some known (for example, uniform) dielectric permeability. The polarization data consist in changes of the polarization between incoming and outcoming monochromatic electromagnetic waves. The basic problem of EPT consists in finding the anisotropic perturbation of the dielectric permeability from the polarization data.

Problem 1 arises as a problem of EPT (with uniform background dielectric permeability) in the framework of the following reduction (see [V.Sharafutdinov 1994], [R.Novikov, V.Sharafutdinov 2007]):
$n = 3$,

$$A(x, \theta) = -\pi_\theta f(x) \pi_\theta, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \tag{15}$$

where f is $M_{3,3}$-valued function describing the anisotropic perturbation of the dielectric permeability tensor; by some physical arguments f must be skew-Hermition, $f_{ij} = -\bar{f}_{ji}$,

$$\pi_\theta \in \mathcal{M}_{3,3}, \quad \pi_{\theta,ij} = \delta_{ij} - \theta_i \theta_j;$$

S for equation (1) with A given by (15) describes the polarization data, but, in general, it can not be given explicitly already.

In this case S on $T\mathbb{S}^2$ does not determine f on $\mathbb{R}^3$ uniquely, in general, [R.Novikov, V.Sharafutdinov 2007] (in spite of the fact that $dim\, T\mathbb{S}^2 = 4 > dim\, \mathbb{R}^3 = 3$), in particular, if

$$f_{11} = f_{22} = f_{33} \equiv 0, \tag{16}$$

$$f_{12}(x) = \partial u(x)/\partial x_3, \quad f_{13}(x) = -\partial u(x)/\partial x_2, \quad f_{23}(x) = \partial u(x)/\partial x_1,$$

$$f_{21} = -f_{12}, \quad f_{31} = -f_{13}, \quad f_{32} = -f_{23},$$

where u is a real smooth compactly supported function, then $S \equiv Id$ on $T\mathbb{S}^2$.

On the other hand, a very natural additional physical assumption is that f is an imaginary-valued symmetric matrix: $f = -\bar{f}$, $f_{ij} = f_{ji}$. According to [R.Novikov 2009], in this case

$$S \text{ on } \Lambda \text{ uniquely determines } f, \text{ at least, if } f \text{ is sufficiently small,} \tag{17}$$

where Λ is an appropriate $3d$ subset of $T\mathbb{S}^2$, for example,

$$\Lambda = \cup_{i=1}^{6} \Gamma_{\omega^i}, \quad \Gamma_{\omega^i} = \{\gamma = (x, \theta) \in T\mathbb{S}^2 : \ \theta\omega^i = 0\}, \tag{18}$$

$$\omega^1 = e_1, \quad \omega^2 = e_2, \quad \omega^3 = e_3,$$

$$\omega^4 = (e_1 + e_2)/\sqrt{2}, \quad \omega^5 = (e_1 + e_3)/\sqrt{2}, \quad \omega^6 = (e_2 + e_3)/\sqrt{2},$$

where e_1, e_2, e_3 is the basis in $\mathbb{R}^3$. In addition, this determination is based on a convergent iterative reconstruction algorithm.

II. Differential geometry:

$$A(x,\theta) = \sum_{j=1}^{d} \theta_j a_j(x), \quad x \in \mathbb{R}^d, \quad \theta = (\theta_1, \dots, \theta_d) \in \mathbb{S}^{d-1}, \quad (19)$$

where a_j are sufficiently regular $\mathcal{M}_{n,n}$-valued functions on $\mathbb{R}^d$ with sufficient decay as $|x| \to \infty$. In this case equation (1) describes the parallel transport of the fibre in the trivial vector bundle with the base $\mathbb{R}^d$ and the fibre $\mathbb{C}^n$ and with the connection $a = (a_1, \dots, a_d)$ along the Euclidean geodesics in $\mathbb{R}^d$; in addition, $S(\gamma)$ for fixed $\gamma \in T\mathbb{S}^{d-1}$ is the operator of this parallel transport along γ (from $-\infty$ to $+\infty$ on γ).
In this case Problem 1 is an inverse connection problem. The determination in this problem is considered modulo gauge transformations.

The results of [R.Novikov 2002 a] on this problem include:
global uniqueness and reconstruction results in dimension $d \geq 3$ (based on local 2D - reconstructions based on solving Riemann conjugation problems and on the layer by layer reconstruction approach);
counter examples to the global uniqueness in dimension $d = 2$ (using Ward's solitons for an integrable chiral model in 2+1 dimensions).
In connection with the inverse connection problem along non-Euclidean geodesics we refer to [V. Sharafutdinov 2000], [G. Paternain 2013], [C. Guillarmou, G. Paternain, M. Salo, G. Uhlmann 2016] and references therein.

III. Theory of the Yang-Mills fields:
A. The aforementioned inverse connection problem arises, in particular, in the framework of studies on inverse problems for the Schrödinger equation

$$\sum_{j=1}^{d} -\left(\frac{\partial}{\partial x_j} + a_j(x)\right)^2 \psi + v(x)\psi = E\psi \qquad (20)$$

in the Yang-Mills field $a = (a_1, \ldots, a_d)$ at $E \to +\infty$ (see [R.Novikov 2002 a].
B. Integration of the self-dual Yang-Mills equations by the inverse scattering method (see [S.Manakov, V.Zakharov 1981], [R.Ward 1988].

Actually, Problem 1 for

$$A(x,\theta) = a_0(x)+\theta_1 a_1(x)+\theta_2 a_2(x),\;\; x=(x_1,x_2)\in\mathbb{R}^2,\;\; \theta=(\theta_1,\theta_2)\in\mathbb{S}^1, \tag{21}$$

with $\mathcal{M}_{n,n}$-valued a_0, a_1, a_2 (and some linear relation between a_1 and a_2) was considered for the first time in [S.Manakov, V.Zakharov 1981] in the framework of integration by inverse scattering method of the evolution equation

$$(\chi^{-1}\chi_t)_t = (\chi^{-1}\chi_z)_{\bar{z}}, \tag{22}$$

where $t, z, \bar{z}$ in (22) denote partial derivatives with respect to t, $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ and where χ is $SU(n)$-valued function. Equation (22) is a (2+1)-dimensional reduction of the self-duel Yang-Mills equations in 2+2 dimensions.

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