Radon 100 March 30, 2017

Inverse Transport Problems (Generalized) Stability Estimates

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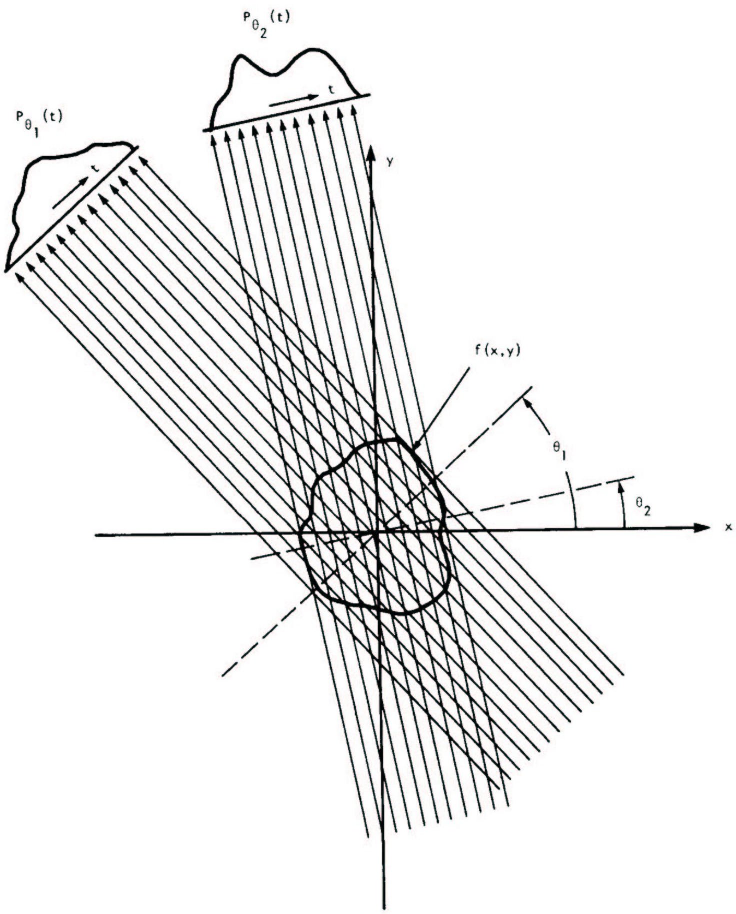
Data Acquisition in CT-scan



$a(x)$ is unknown absorption coefficient. For each line in the plane , the measured ratio $\boxed{u_{\text{out}}(s,\theta)/u_{\text{in}}(s,\theta)}$ is equal to:

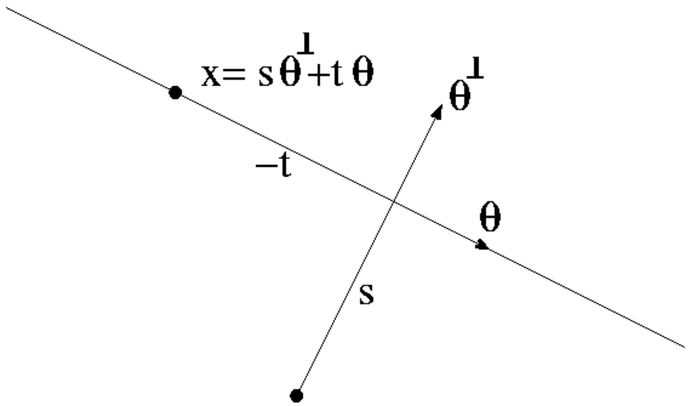
exp $\left(-\int_{\text{line(s},\theta)} a(x)dl\right)$. s =line-offset.

The X-ray density $u(x,\theta)$ solves the transport equation

$$\boxed{\boldsymbol{\theta}\cdot\nabla u(x,\theta)+a(x)u(x,\theta)=0.}$$

Here, x is position and $\boldsymbol{\theta} =$ $(\cos\theta, \sin\theta)$ direction.

Radon transform.



We define the Radon transform

$$Ra(s,\theta) = R_\theta a(s) = \int_{\mathbb{R}} a(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta})dt$$

for $s \in \mathbb{R}$ and $\boldsymbol{\theta} \in S^1$.

Note the redundancy:
$Ra(-s,\theta+\pi) = Ra(s,\theta)$.

Introduce the adjoint operator and the Hilbert transform

$$R^* g(x) = \int_0^{2\pi} g(x\cdot\boldsymbol{\theta}^\perp,\theta)d\theta, \qquad Hf(t) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(s)}{t-s} ds.$$

Then we have (e.g. in the L^2-sense) the reconstruction formula

$$Id = \frac{1}{4\pi} R^* \frac{\partial}{\partial s} H R = \frac{1}{4\pi} R^* H \frac{\partial}{\partial s} R.$$

Stability Estimate.

Stability estimates for the Radon transform may be obtained as follows. For X a bounded domain in $\mathbb{R}^2$ and a supported in $\bar{X}$, we have for $C = C(X)$,

$$\boxed{\frac{1}{C}\|a\|_{L^2(X)} \le \|Ra\|_{H^{\frac{1}{2}}(\mathbb{R}\times S^1)} \le C\|a\|_{L^2(X)}}$$

with $\|g\|_{H^{\frac{1}{2}}(\mathbb{R}\times S^1)} = \|(1 - d_s^2)^{\frac{1}{2}} g\|_{L^2(\mathbb{R}\times S^1)}$.

Note the stability estimate is directly for $a(x)$, not for the physical measurements based on the transport solution $u(x, \theta)$.

Scattering Scattering

As we heard in previous talks, several applications have at their core an inverse Radon transform: CT, SPECT, PET.

Neglected so far: scattering. Scattering is typically not very informative (no contrast) but it is there. Its reconstruction helps with that of other coefficients.

[Courdurier, Monard, Osses, Romero. Simultaneous source and attenuation reconstruction in SPECT using ballistic and single scattering data 2015;
Phys Med Biol. 2011; Review and current status of SPECT scatter correction. Hutton B.F., Buvat I, Beekman F.J.
Talk here in, e.g., MS17 by Herbert Egger, Vadim Markel]

Kinetic Model

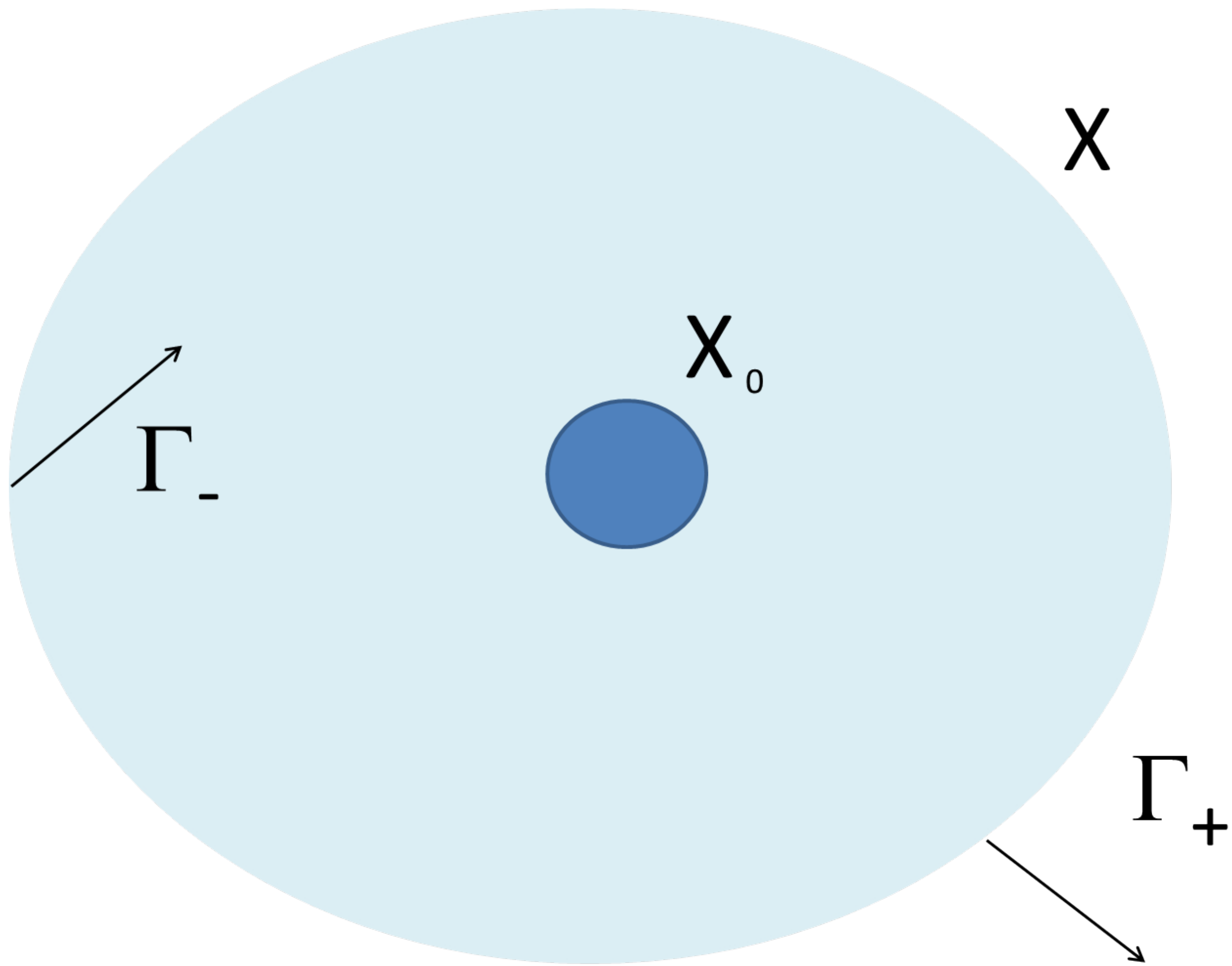
Consider the transport equation for convex bounded $X \subset \mathbb{R}^n$; $V = \mathbb{S}^{n-1}$:

$$v \cdot \nabla u + \sigma(x,v)u = \int_V k(x,v',v)u(x,v')dv', \quad (x,v) \in X \times V$$
$$u = g, \quad (x,v) \in \Gamma_-; \qquad \Gamma_\pm = \{(x,v) \in \partial X \times V, \pm v \cdot n(x) > 0\}.$$

Inverse Problem: Reconstruct (information on) $\sigma(x,v)$ and $k(x,v',v)$ from knowledge of the Albedo operator $\underline{\mathcal{A}[\sigma,k] : g = u_{\Gamma_-} \mapsto \mathcal{A}g = u_{\Gamma_+}}$.

(Many related inverse problems where the albedo operator is partially known, for instance, as in Optical Tomography [Arridge Schotland IP 09; B. IP 09], isotropic sources and angularly averaged measurements. Not discussed further here.)

Geometry of transport problem



Inverse Problems with full measurements

Consider the inverse problem with $0 \leq k, \sigma$ bounded and $\int_V k dv' \leq \sigma$. Then $\mathcal{A}$ is defined from $L^1(\Gamma_-; d\xi)$ to $L^1(\Gamma_+; d\xi)$ with $d\xi = |n \cdot v| d\mu(x) dv$ and its Schwartz kernel $\alpha(x, v; x', v')$ admits the following decomposition into ballistic, single scattering, and multiple scattering contributions:

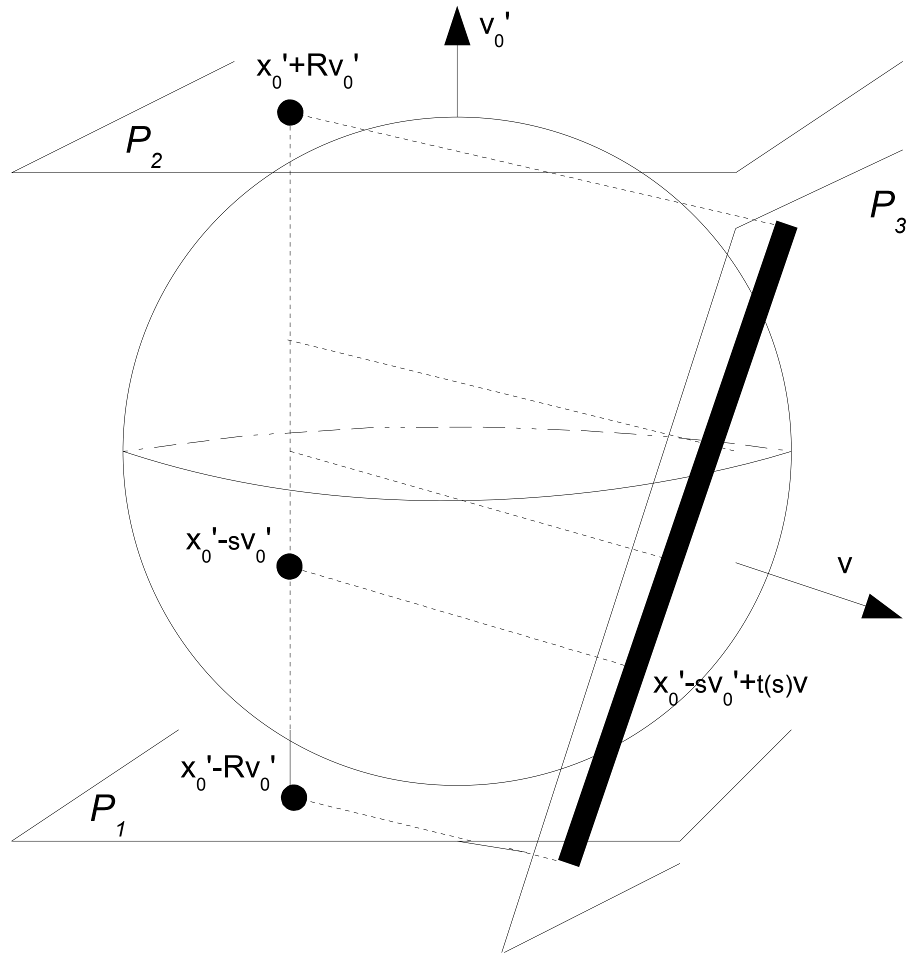
$$\alpha(x, v; x', v') = \alpha_0(x, v; x', v') + \alpha_1(x, v; x', v') + \alpha_2(x, v; x', v'),$$

with, introducing $E(x, x') = \exp(-\int_0^{|x-x'|} \sigma(x - s\frac{x-x'}{|x'-x|}, \frac{x-x'}{|x'-x|}) ds)$,

$$\begin{aligned}
\alpha_0 &= \omega E(x, x') \delta_{\{v'\}}(v) \delta_{\{x' + \tau_+ v'\}}(x), \qquad \omega = \frac{|n(x') \cdot v'|}{n(x) \cdot v} \\
\alpha_1 &= \omega \int_0^{\tau_+(x,v)} E(x' + tv', x') k(x' + tv', v', v) E(x, x' + tv') \delta_{\{x' + tv' + \tau_+ v\}}(x) dt \\
\alpha_2 & \quad \text{is a function}
\end{aligned}$$

α_0 is more singular than α_1, which is more singular than α_2 when $n \geq 3$.

Geometry of singularities



Inverse transport theory

$$v \cdot \nabla u + \sigma(x,v)u = \int_V k(x,v',v)u(x,v')dv', \quad (x,v) \in X \times V$$
$$u = g, \quad (x,v) \in \Gamma_-; \qquad\qquad \Gamma_\pm = \{(x,v) \in \partial X \times V, \pm v \cdot n(x) > 0\}.$$

Theorem [Choulli Stefanov 1999]

For $n \geq 2$, knowledge of $\mathcal{A}$ implies that of α_0 on $\Gamma_+ \times \Gamma_-$, which uniquely determines $\sigma = \sigma(x)$ by inverse Radon transform.

For $n \geq 3$, knowledge of $\mathcal{A}$ implies that of α_1 on $\Gamma_+ \times \Gamma_-$, which uniquely determines $k(x,v',v)$.

In dimension $n = 2$, reconstruction of $k(x,v',v)$ is known only under smallness assumption [Stefanov Uhlmann 03].

Stability of the reconstructions

The reconstructions are stable in the following sense.

Theorem [B. Jollivet 09] In dimension $n \geq 2$, we have

$$\left| \int_{\mathbb{R}} (\sigma - \tilde{\sigma})(x + tv)dt \right| \sim \left| (E - \tilde{E})(x, x') \right| \leq \varepsilon := \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1(\Gamma_-);L^1(\Gamma_+))}.$$

In dimension $n \geq 3$, we have $\int_V \int_{\mathbb{R}} |EkE - \tilde{E}\tilde{k}\tilde{E}|(x + tv, v', v)dtdv \leq C\varepsilon$, which implies:

$$\int_V \int_{\mathbb{R}} |k - \tilde{k}|(x + tv, v', v)dtdv \leq C\Big(\sup |(E - \tilde{E})| + \varepsilon\Big).$$

The reconstructions of the Radon transform of σ, and k (once σ is reconstructed) are Lipschitz-stable with respect to errors in the measurements.

Anisotropic $\sigma(x,v)$

When $\sigma = \sigma(x,v)$ and $k = 0$, it is clearly not possible to uniquely reconstruct σ from its line integrals.
Then (σ, k) and (σ', k') are called gauge equivalent if there is $\phi(x,v)$ such that $0 < \phi_0 \le \phi(x,v) \le \phi_0^{-1}$, $|v \cdot \nabla\phi|$ bounded, $\phi = 1$ on $\partial X \times V$, and

$$\sigma' = \sigma - v \cdot \nabla \log \phi, \qquad k'(x,v',v) = \frac{\phi(x,v)}{\phi(x,v')} k(x,v',v).$$

Note that u and ϕu then solve the same transport equation. Let $< \sigma, k >$ be the class of equivalence. Then:

Theorem [McDowall, Stefanov, Tamasan 10] In dimension $n \ge 3$, $\mathcal{A}$ uniquely determines $< \sigma, k >$ and

$$\|\sigma' - \tilde{\sigma}\|_\infty + \|k' - \tilde{k}\|_1 \le C \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}, \qquad (\sigma', k') \in < \sigma, k > .$$

Corollary: when $k \ge k_0 > 0$, $k(x,v',v) = k(x,v,v')$ and $\sigma(x,-v) = \sigma(x,v)$, then (σ, k) is uniquely determined.

Stability inverse transport

In summary: We have uniqueness and stability results for classes of equivalence. Full uniqueness results *must* follow from additional prior information.

Coefficient reconstruction estimates then follow by regularity assumptions and interpolations. For instance, for $\sigma=\sigma(x)$, define for some $r>0$ and $M>0$,

$$\mathcal{M}=\Big\{(\sigma,k)\in C^0(\bar{X})\times C^0(\bar{X}\times V\times V)|\sigma\in H^{\frac{d}{2}+r}(X),\|\sigma\|_{H^{\frac{d}{2}+r}(X)}+\|k\|_\infty\leq M\Big\}$$

Then [Wang 99, B.Jollivet 08,09]

$$\|\sigma-\tilde{\sigma}\|_{H^s(X)}\leq C\|\mathcal{A}-\tilde{\mathcal{A}}\|^{\kappa}_{\mathcal{L}(L^1)},\qquad -\frac{1}{2}\leq s<\frac{d}{2}+r \quad\text{and}\quad \kappa=\frac{d+2(r-s)}{d+1+2r}$$

$$|k-\tilde{k}\|_{L^1(X\times V\times V)}\leq\|\mathcal{A}-\tilde{\mathcal{A}}\|^{\kappa'}_{\mathcal{L}(L^1)}\big(1+\|\mathcal{A}-\tilde{\mathcal{A}}\|^{1-\kappa'}_{\mathcal{L}(L^1)}\big),\quad \kappa'=\frac{2(r-r')}{d+1+2r},\quad 0<r'<r.$$

Numerical Inverse Transport

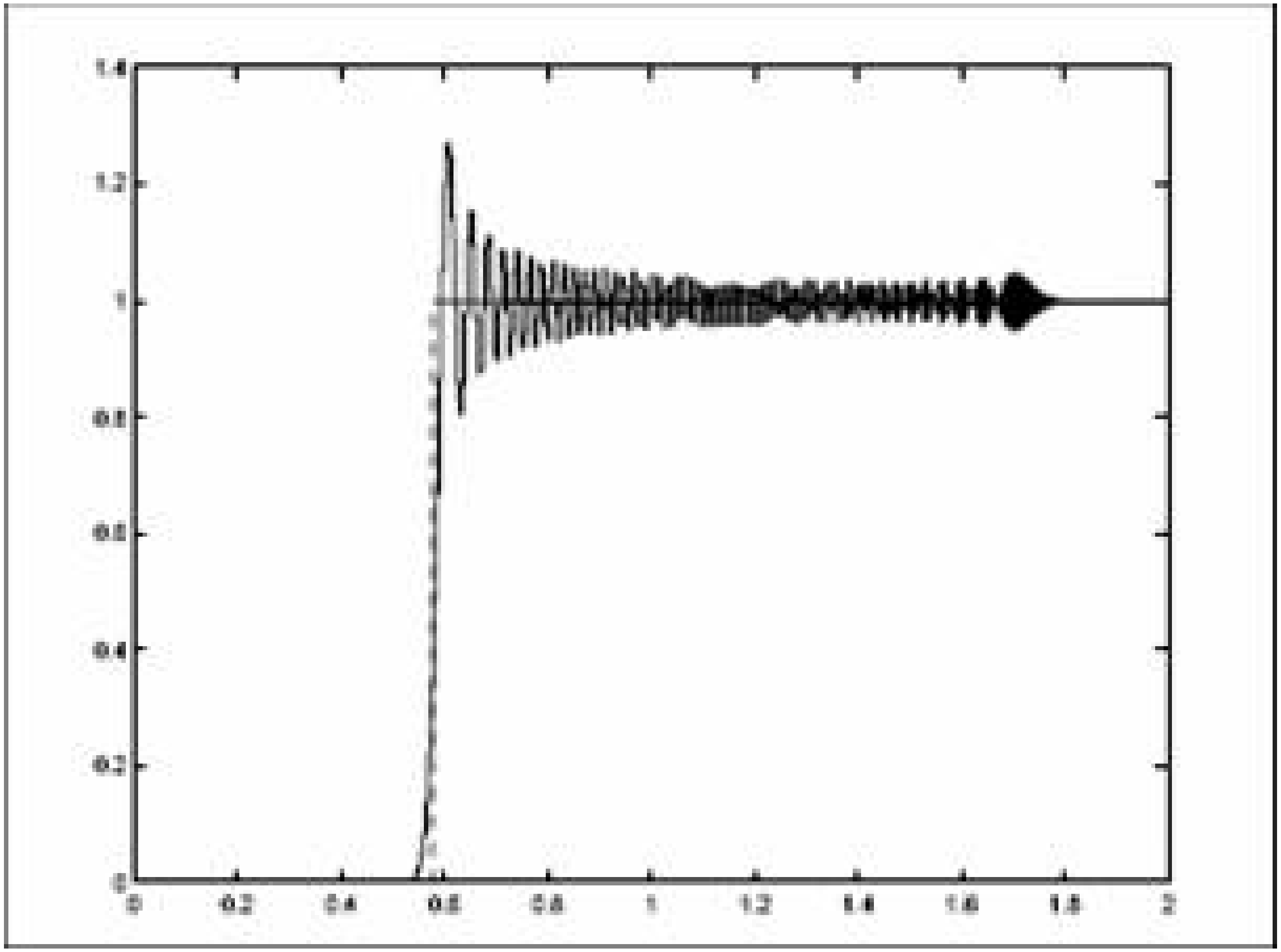
Let us briefly consider numerical inversions [B. Monard, JCP 2010]. Solving

$$\Big(v \cdot \nabla + \sigma(x)\Big)u = 0,$$

accurately is actually **difficult** on a grid. $v \cdot \nabla = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad d = 2.$

Many standard methods will not accurately propagate singularities, which are the reason why inverse transport is reasonably well-posed.

For instance, Diamond Discretization (second-order stable L^2-isometry) displays a lot of numerical dispersion.



Here is how a step function is translated in the absence of attenuation [B. Calcolo 2001]. (The exact solution is another step function!)

A specific method to propagate singularities in numerical transport:

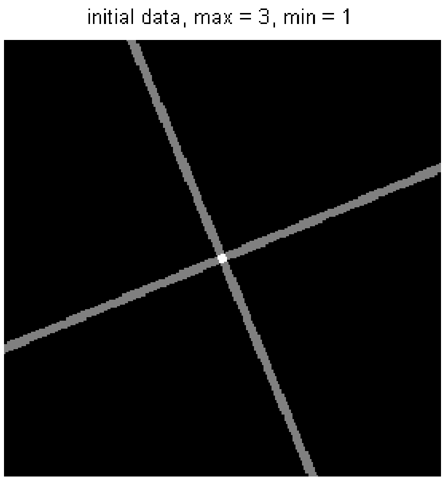
We solve the above equation accurately by using a slanting algorithm: the whole domain is slanted so that any direction becomes one of the axes of a Cartesian grid after slanting.

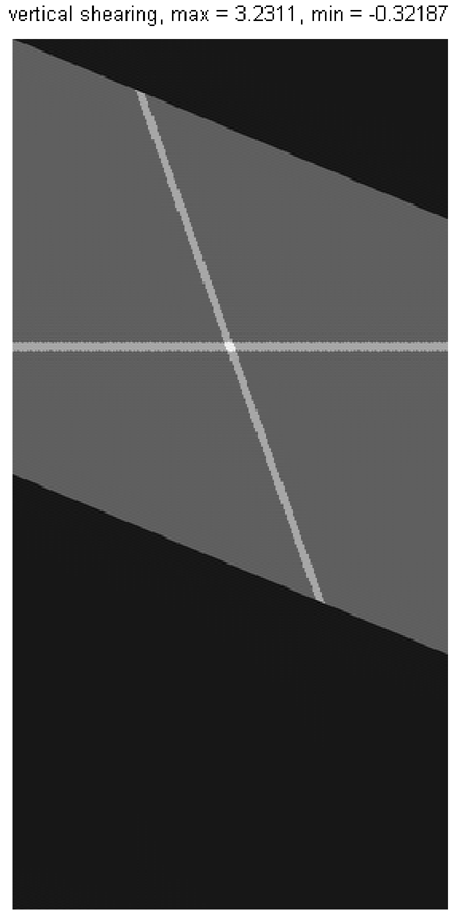
After slanting, we need to solve

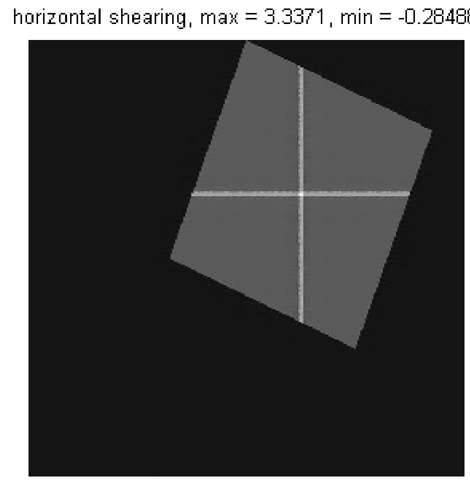
$$\left(\frac{\partial}{\partial x} + \tilde{\sigma}(x)\right)\tilde{u} = 0,$$

which is Cartesian friendly. Slanting is done with spectral accuracy using FFT-type algorithms.

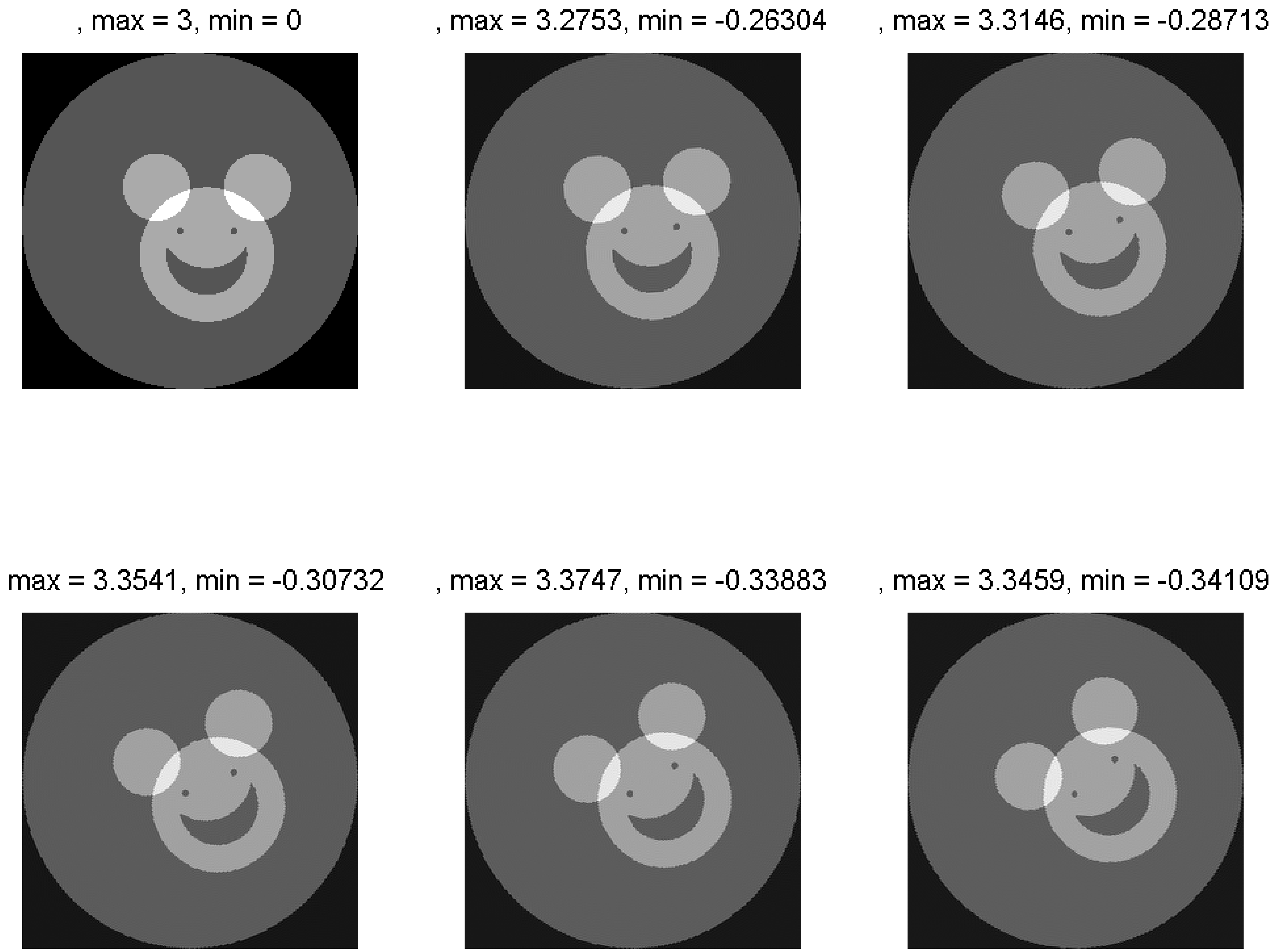
Slanting and rotating







Rotating 256×256



Accounting for small angular diffusion

In the rotated variables, it is relatively straightforward to add an approximation (paraxial approximation) of angular diffusion (noise contribution):

$$\Big(\frac{\partial}{\partial x} + \tilde{\sigma}(x) - \tilde{\varepsilon}(x)\frac{\partial^2}{\partial y^2}\Big)\tilde{u} = 0,$$

This becomes a parabolic operator that may be solved (implicitly since $\Delta x = \Delta y$ is the pixel size) rapidly and robustly.

Numerical inversion

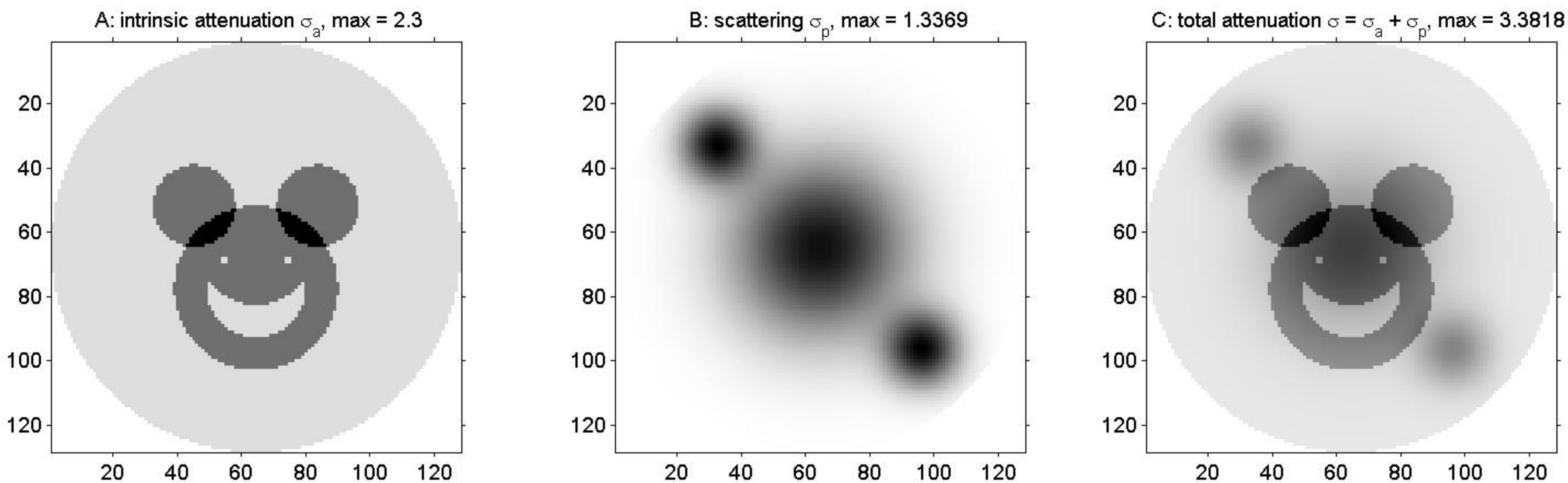
$$v \cdot \nabla u + \sigma(x,v)u = \int_V k(x,v',v)u(x,v')dv', \quad (x,v) \in X \times V$$
$$u = g, \quad (x,v) \in \Gamma_-; \qquad \Gamma_\pm = \{(x,v) \in \partial X \times V, \pm v \cdot n(x) > 0\}.$$

(i) From the ballistic component of $\mathcal{A}$, we extract line integrals of the attenuation coefficient $\sigma(x)$. The attenuation coefficient is then reconstructed by inverse Radon transform.

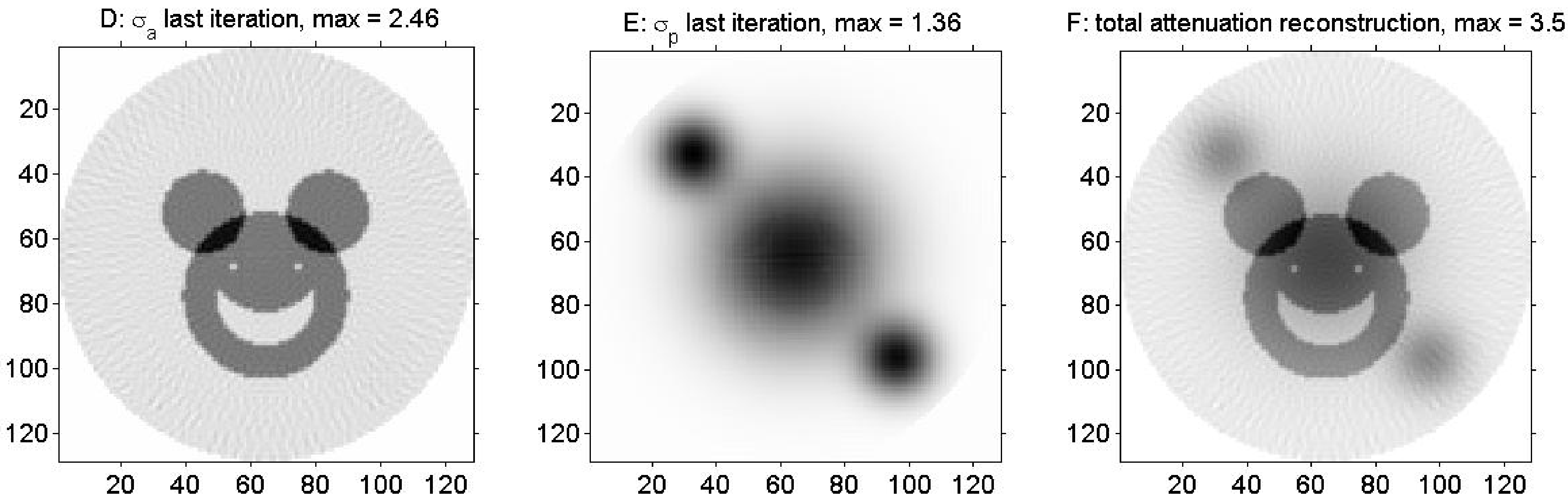
(ii) From the single scattering component of$\mathcal{A}$, we reconstruct the scattering kernel $k(x)$ from a local formula.

(iii) We iterate steps (i) and (ii) until convergence.

Inversion for optical coefficients

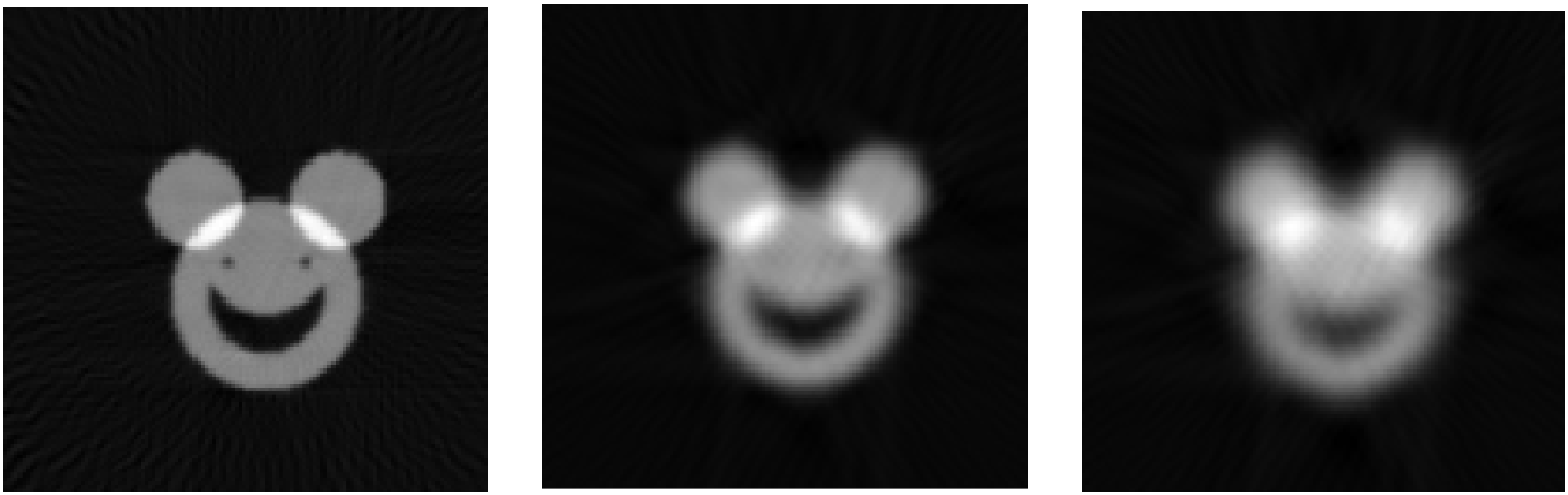


Iterated absorption and scattering reconstruction. Sub-plots A,B,C show the exact parameters.



Sub-plots D and E show the reconstructions of σ_a and σ_p after convergence of the iterative scheme, while sub-plot F shows the reconstruction of the total attenuation.

Inversion with increasing angular diffusion



Diffusion $\varepsilon = 2\ 10^{-4}$; $\varepsilon = 2\ 10^{-3}$; $\varepsilon = 5\ 10^{-3}$. Classical inversion algorithms are very blurry.

What did we do right?

Uniqueness: ✓

Stability: ✓

Numerical validation: ✓

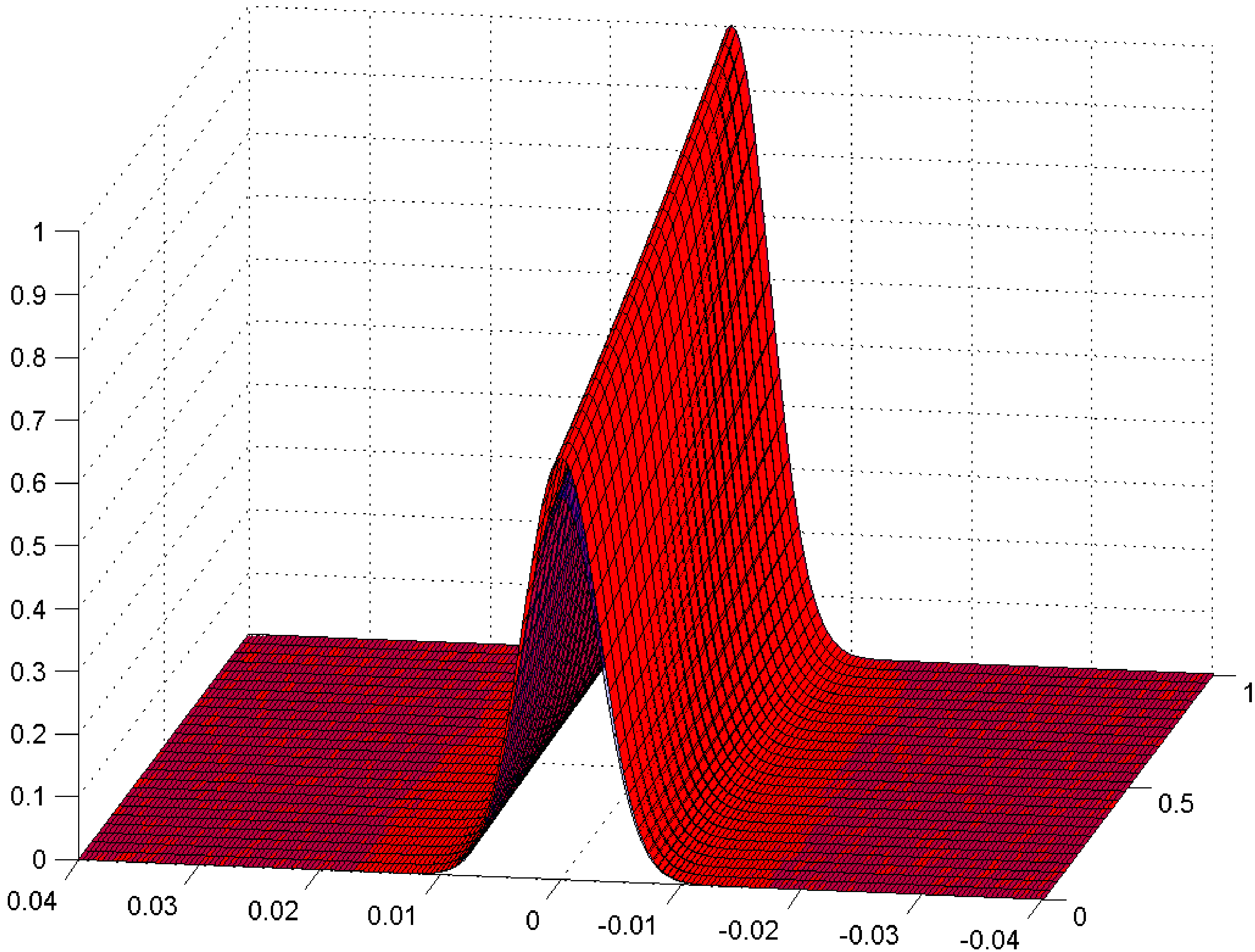
What are we missing?

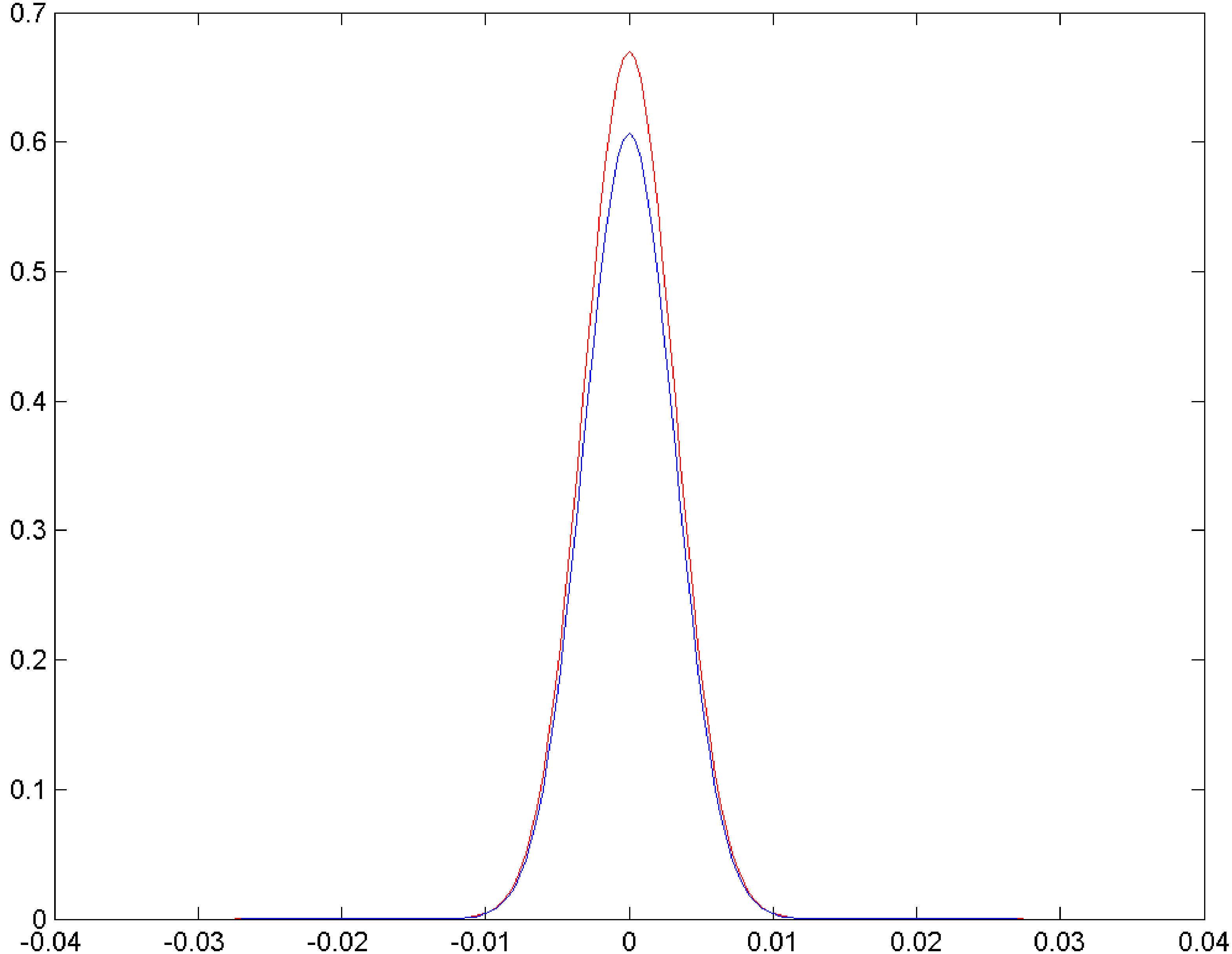
Stability estimates of the form

$$\|\sigma' - \tilde{\sigma}\|_\infty + \|k' - \tilde{k}\|_1 \leq C\|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)},$$

are useful if the *right-hand side is small when noise is small*. This is a problem in many practical settings, where noise may not be modeled by a variation of albedo operators or where $\|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}$ is not small.

There is nothing wrong (mathematically) with the stability estimates. But they may not be the right tool to assess data errors and, hence, reconstruction errors.

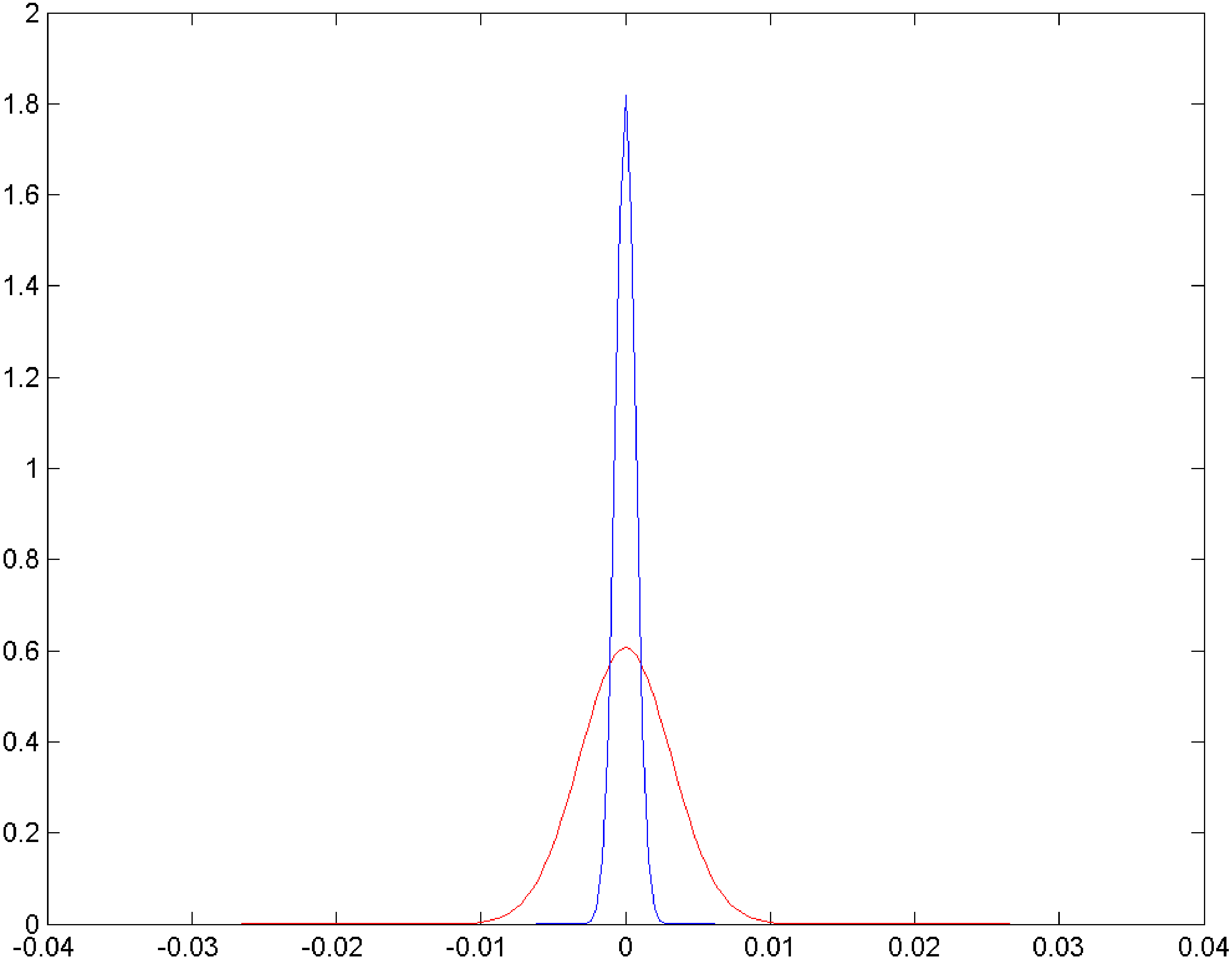

1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
0
0.5
1

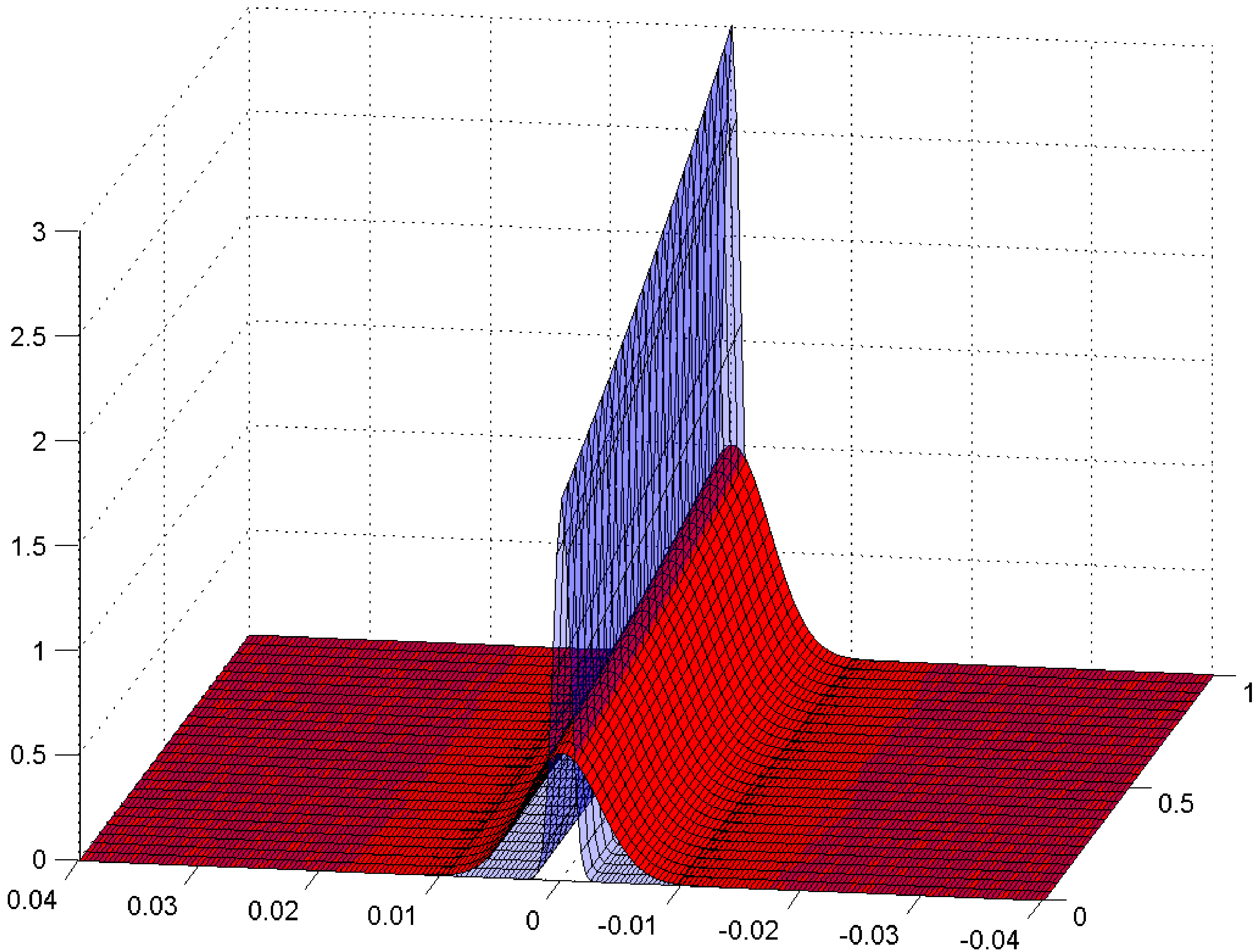

0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.04
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
0.04

Detector noise models

The preceding example is a good application of the above stability estimates: small errors in the measurements translate into small errors in the reconstructions.

Now consider blurring at the detector level.

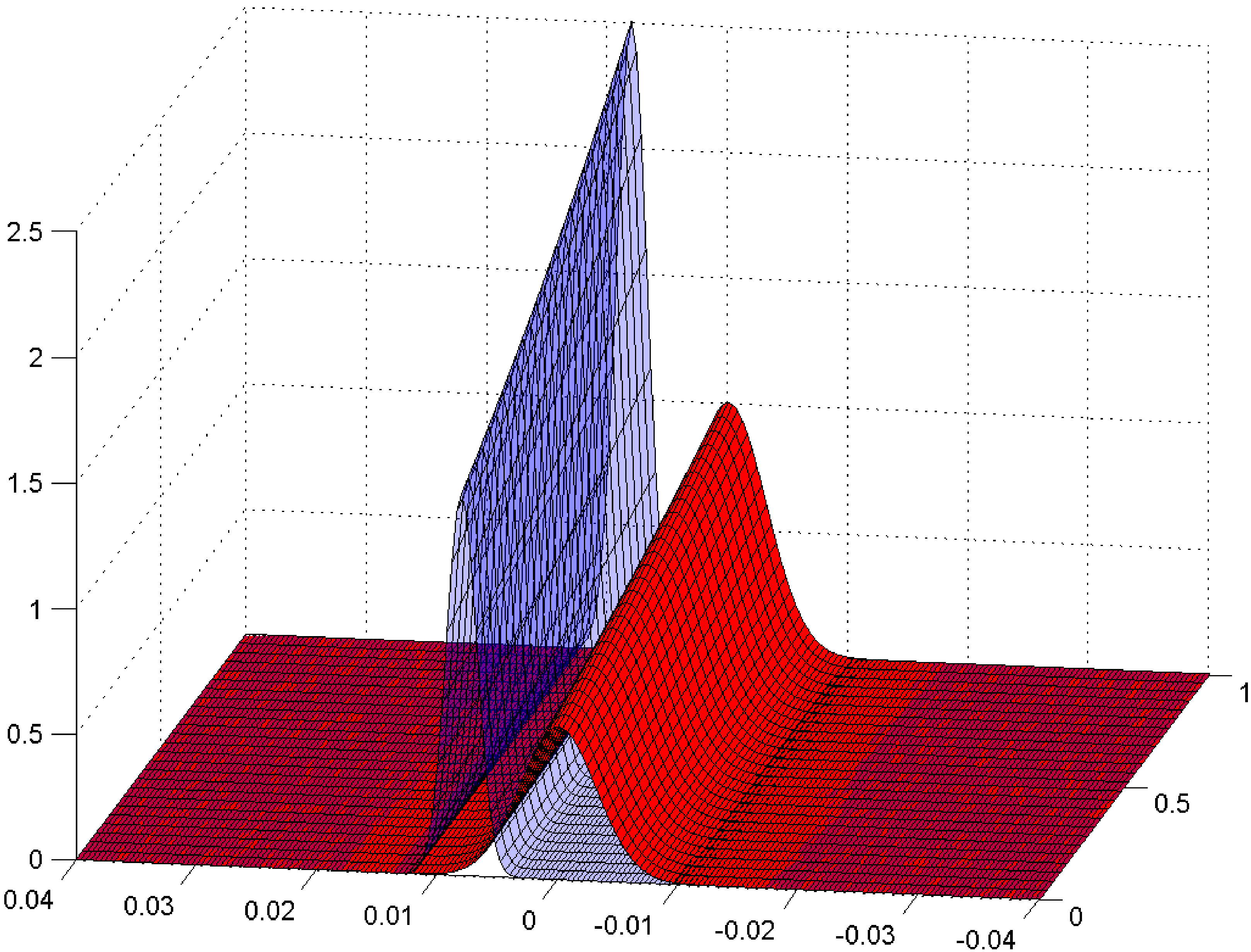

2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.04
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
0.04

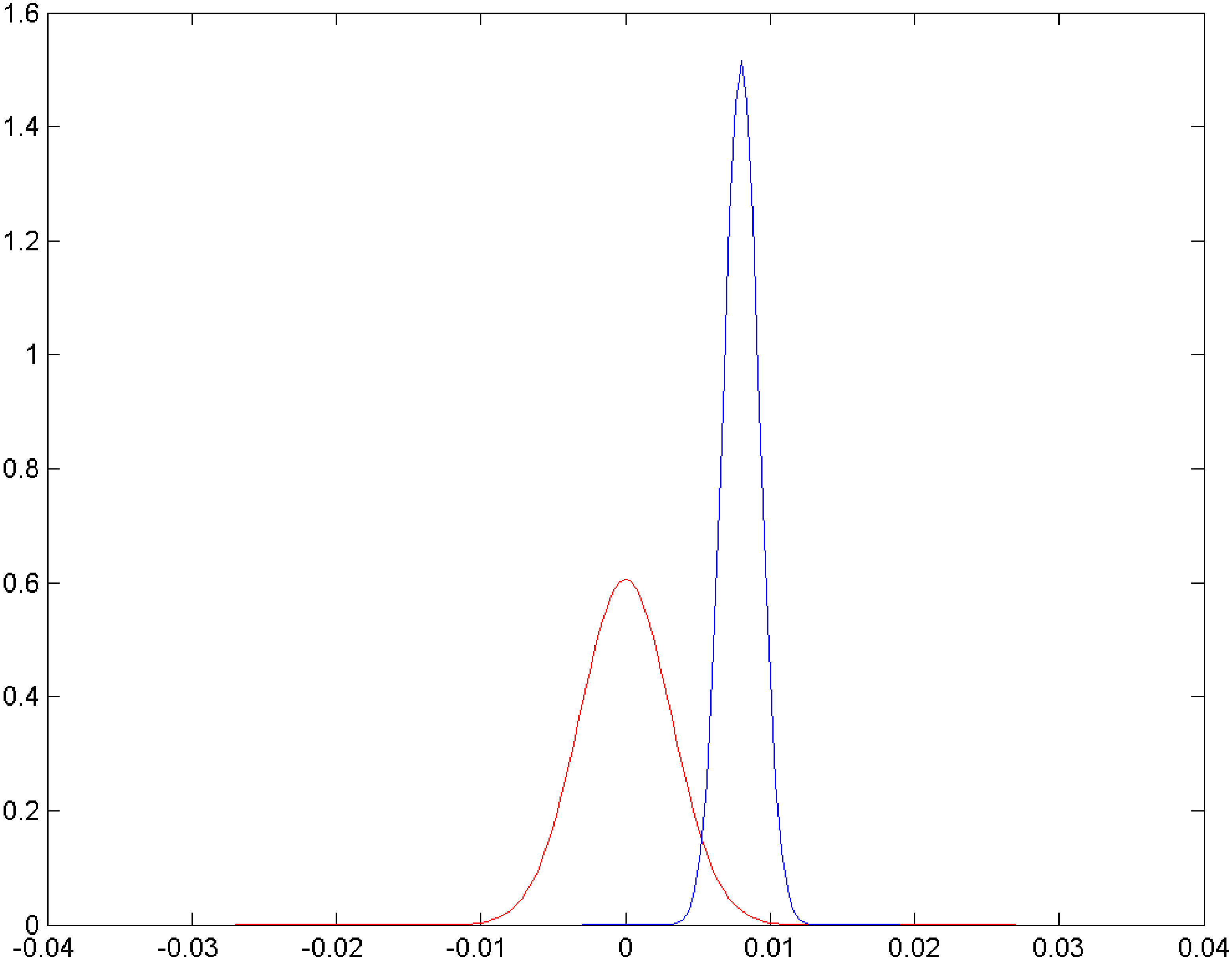

3
2.5
2
1.5
1
0.5
0
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
0
0.5
1

Detector and source noise models

In the preceding example, detector noise moves available data away from the range of the albedo operator. Any “projection” onto that range would result in potentially catastrophic errors levels.

Now consider errors at the source level. Sources are never perfectly in the predicted shape and location.


2.5
2
1.5
1
0.5
0
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
0
0.5
1

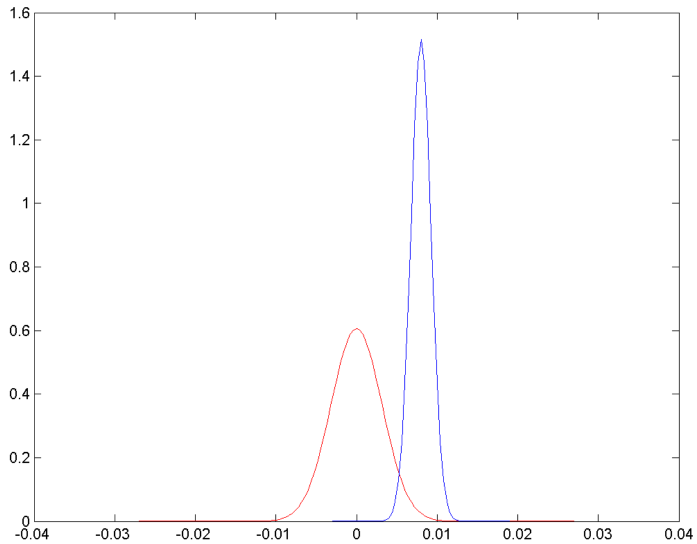

1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.04
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
0.04

Source noise models

Here again, the errors move available data away from the range of the albedo operator. Again, 'projection" onto that range would result in catastrophic errors levels as the L^1 norm between the "blue" and "red" graphs is large.

Yet, small blurring at the detector level or small mis-alignment of the source term *typically do not* result is catastrophic reconstructions for an inverse transport / inverse Radon problem that is not irremediably ill-posed. **How should one model this?**

Modeling of errors and estimates of their influence



We need a metric ensuring that small blurring and small mis-alignment result in small errors. We then wish to assess the damage caused on the reconstruction of the optical coefficients.

Noise modeling

Imposing $\|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}$ small is too constraining. Let $\mathcal{A}_\varepsilon$ with Schwartz kernel $\varphi_\varepsilon *_{(x,v)} \alpha(x,v;x',v')$ corresponding to detector blurring. Then $\|\mathcal{A}-\mathcal{A}_\varepsilon\|_{\mathcal{L}(L^1)} \sim 2$ *independent* of ε. We need a *weaker* metric on $u_{|\Gamma_+}$.

Let μ,ν be finite Radon measures. The Radon distance

$$\rho(\mu,\nu) = \sup\left\{ \int_Y f(y)(\mu(dy)-\nu(dy)), \quad f: Y \mapsto [0,1] \text{ continuous } \right\}$$

is similarly too constraining.

We consider the family of (1-) Wasserstein distances

$$\boxed{W_{1,\kappa}(\mu,\nu) = \sup_{\|\phi\|_\infty \le 1,\ \mathrm{Lip}(\phi)\le\kappa} \langle \phi, \mu-\nu\rangle.}$$

$\phi = \pm 1$ controls the difference between, e.g., $\mu(\Gamma_+)$ and $\nu(\Gamma_+)$.
An increase in κ indicates an improved confidence in the measurement detectors.

Modeling of both noise and source errors

Recall the Wasserstein distance

$$W_{1,\kappa}(\mu,\nu) = \sup_{\|\phi\|_\infty \le 1,\ \mathrm{Lip}(\phi)\le\kappa} \langle \phi, \mu-\nu\rangle.$$

Let g be a non-negative source/detector with a support in the h vicinity of $(x_0, v_0) \in \Gamma_\pm$ integrating to $1+\varepsilon$. Then $\underline{W_{1,\kappa}(g, \delta_{x_0}(x)\delta_{v_0}(v)) \le \kappa h + |\varepsilon|}$.

For g an exact source with $\|g\|_{L^1(\Gamma_-,d\xi)} = 1$, we consider two approximations g_j of g used to probe the domain X. Solutions on Γ_+ are thus $\mathcal{A}_j g_j$. Let ϕ be a (test) function in $L^\infty(\Gamma_+)$ such that $\|\phi\|_{L^\infty(\Gamma_+)} \le 1$. Then we define the measurement ("projected" onto that test function)

$$m \equiv m_{g,\phi} = \varepsilon_j + \langle \phi, \mathcal{A}_j g_j\rangle, \qquad j = 1,2,$$

where ε_j is a measurement error for such a projection, which is small for small detector blurring and small model errors.

Estimates for Wasserstein distances

Recall the error $m \equiv m_{g,\phi} = \varepsilon_j + \langle \phi, \mathcal{A}_j g_j \rangle$. We assume a source error:

$$\delta_j = \sup_{g \in L^1(\Gamma_-)} W_{1,\kappa}(g, g_j), \qquad \delta = \delta_1 + \delta_2,$$

the maximal error in the experimental setting to approximate probe g.

Theorem. [B. Jollivet '17] Let σ, k Lipschitz. Then for σ:

$$|E_1(x_0, y_0) - E_2(x_0, y_0)| \leq C\Big(\Big(\frac{\varepsilon + \delta}{\kappa}\Big)^{\frac{n-1}{n}} \vee (\varepsilon + \delta)\Big).$$

For the reconstruction of k we have for $n \geq 4$

$$\int_V \int_0^{\tau_+(x_0,v_0)} \Big|\tilde{E}_1 k_1 - \tilde{E}_2 k_2\Big|(x_0 + s v_0, v_0, v) ds dv \leq C\Big(\Big(\frac{\varepsilon + \delta}{\kappa}\Big)^{\frac{1}{2}} \vee (\varepsilon + \delta)\Big).$$

with instead $\left(\frac{\varepsilon+\delta}{\kappa}\right)^{\frac{1}{2}}(1 + \sqrt{|\ln(\frac{\varepsilon+\delta}{\kappa})|}) \vee (\varepsilon + \delta)$ in $n = 3$.

Generalized Stability Estimates

Theorem. For $\mathfrak{e}_b(\varepsilon,\delta)$ and $\mathfrak{e}_s(\varepsilon,\delta)$ the respective above r.h.s, we find

$$\|\tau(\tilde{\sigma}_1-\tilde{\sigma}_2)\|_\infty \le C\mathfrak{e}_b(\varepsilon,\delta) \quad \text{and} \quad \|\tilde{k}_1-\tilde{k}_2\|_1 \le C(\mathfrak{e}_b(\varepsilon,\delta)+\mathfrak{e}_s(\varepsilon,\delta)),$$

for appropriate elements in the class of equivalence $<\tilde{\sigma}_j,\tilde{k}_j>=<\sigma_j,k_j>$.

The main features are

(i) Estimates ε are now of the form $(\frac{\varepsilon}{\kappa})^\alpha \vee \varepsilon$.
(ii) For sharp detectors κ is large so that $(\frac{\varepsilon}{\kappa})^\alpha \sim \varepsilon$.
(iii) The coefficients need to be Lipschitz. This is expected as sources are allowed mis-alignment.
(iv) Framework includes setting with a *finite* number of sources (since any source is close to a finite collection of sources).

Elements of derivation

Multiple scattering contributions can be isolated

$$\alpha_m(z_0, v_0, z_{m+1}, v_m) = |\nu(z_{m+1}) \cdot v_m| \int_0^{\tau_+(z_{m+1},v_m)} \int_0^{\tau_-(z_0,v_0)} \int_{X^{m-2}} \Big[\frac{E(z_0,\ldots,z_{m+1})}{\prod_{i=1}^{m-1} |z_i - z_{i+1}|^{n-1}} \times \prod_{i=1}^{m} k(z_i, v_{i-1}, v_i) \Big]_{|z_1 = z_0 - tv_0,\ z_m = z_{m+1} + sv_m,\ v_i = \widehat{z_i - z_{i+1}},\ i=1\ldots m} dt ds dz_2 \ldots dz_{m-1},$$

and their influence on detectors estimated

$$\frac{|\int_{\Gamma_+} \alpha_m(z, v, z'', v'') \phi(z, v) d\xi(z, v)|}{|\nu(z'') \cdot v''|} \le C_m e^{(m+1)\|\tau\sigma_-\|_\infty} \|k\|_\infty^m \eta^{n+m-2},\ 2 \le m \le n$$

Appropriate choices of ϕ then lead to the aforementioned estimates by appropriately concentrating on the support of the ballistic and single scattering contributions.

What did we do right?

Uniqueness: $\checkmark$

Stability: $\checkmark$

Numerical validation: $\checkmark$

Physics based stability: $\checkmark$

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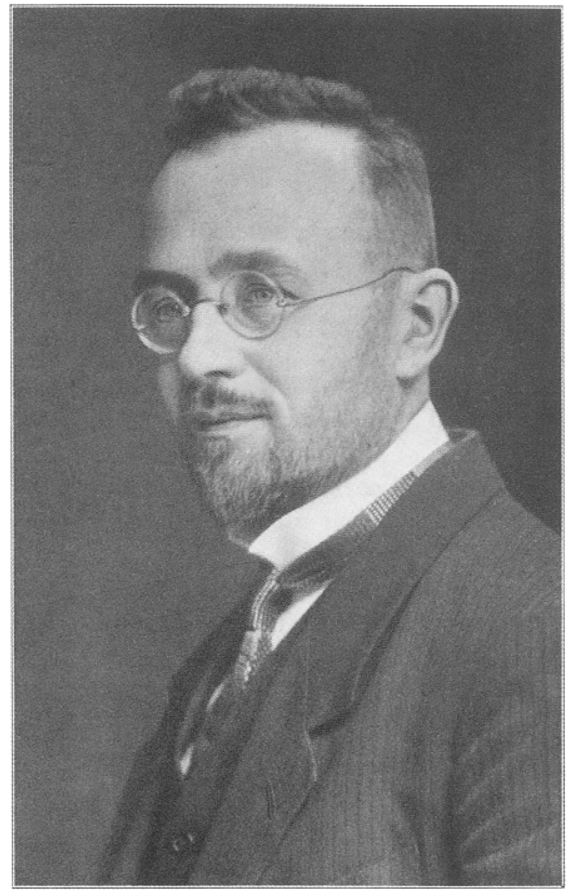

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